INFINITE SQUARE WELL WITH TRIANGULAR INITIAL STATE USING DELTA FUNCTION

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Earlier, we looked at the case of a particle in the infinite square well with an initial wave function that was triangular:

\[ \Psi(x, 0) = \begin{cases} 
  Ax & 0 \leq x \leq a/2 \\
  A(a - x) & a/2 \leq x \leq a 
\end{cases} \]  

(1)

We find \( A \) from normalization:

\[ \int_0^a |\Psi|^2 dx = A^2 a^3/12 = 1 \]  

(2)

so

\[ A = \sqrt{\frac{12}{a^{3/2}}} \]  

(3)

Because the first derivative of \( \Psi(x, 0) \) is discontinuous at \( x = 0 \), we might encounter problems in calculating the second derivative, which we need to find the mean value of the energy if we use integration, since this is

\[ \langle E \rangle = \frac{\hbar^2}{2m} \int_0^a \Psi^*(x, 0) \frac{d^2}{dx^2} \Psi(x, 0) dx \]  

(4)

We can express the first derivative of wave function as a step function \( \theta(x) \):

\[ \frac{d\Psi(x, 0)}{dx} = \begin{cases} 
  A & 0 < x < a/2 \\
  -A & a/2 < x < a 
\end{cases} = -A(2\theta(x - a/2) - 1) \]  

(5)

where

\[ \theta(x) = \begin{cases} 
  1 & x < 0 \\
  0 & x > 0 
\end{cases} \]  

(6)
We’ve seen that the derivative of the step function can be taken as the delta function, so
\[
\frac{d^2\Psi(x,0)}{dx^2} = -2A\delta(x - a/2)
\]  
(7)

Using the delta function directly, we get
\[
\langle H \rangle = -\frac{\hbar^2}{2m} \int_0^a \Psi^* \frac{d^2\Psi}{dx^2} dx = \frac{\hbar^2 A^2}{m} \frac{a}{2} = \frac{6\hbar^2}{ma^2}
\]  
(8)

using $A^2 = 12/a^3$ from above. The final result is the same as that from summing the series as we did earlier.