HARMONIC OSCILLATOR: SCHRÖDINGER'S EXACT SOLUTION

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Post date: 24 Aug 2012.

References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.49.

An example of an exact solution to the time-dependent Schrödinger equation with the harmonic oscillator potential was discovered by Schrödinger himself. It's not pretty, but it looks like this

$$\Psi(x,t)=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{m\omega}{2\hbar}\left(x^{2}+\frac{a^{2}}{2}\left(1+e^{-2i\omega t}\right)+\frac{i\hbar t}{m}-2axe^{-i\omega t}\right)\right] \tag{1}$$

where a is a constant.

Verifying that this is indeed a solution is a long slog on paper, but using Maple it's not too difficult. We work out the two sides of the Schrödinger equation and show they are equal.

For the left, we have

$$-\frac{\hbar^{2}}{2m}\frac{\partial^{2}\Psi}{\partial x^{2}}+\frac{1}{2}m\omega^{2}x^{2}\Psi \tag{2}$$

and for the right

$$i\hbar\frac{\partial\Psi}{\partial t} \tag{3}$$

Both of these come out to a rather hideous expression, but on dividing the left by the right and simplifying, we get 1, so the two expressions are in fact the same.

To calculate the square modulus of the wave function, we can use Maple's 'abs' function which calculates the modulus of a complex number. The result is

$$|\Psi(x,t)|^2 = \sqrt{\frac{m\omega}{\pi\hbar}} e^{-m\omega\left(x^2+a^2(1+\cos(2\omega t))/2-2ax\cos(\omega t)\right)/\hbar} \tag{4}$$

$$= \sqrt{\frac{m\omega}{\pi\hbar}} e^{-m\omega(x-a\cos(\omega t))^2/\hbar} \tag{5}$$

Thus the wave packet is a Gaussian curve whose peak oscillates about $x = 0$ with period $2\pi/\omega$ and amplitude a.

At $t = 0$, this becomes:

$$= |\Psi(x,0)|^2 = \sqrt{\frac{m\omega}{\pi\hbar}} e^{-m\omega(x-a)^2/\hbar} \tag{6}$$

so it is a Gaussian bell-shaped curve centred at $x = a$. That is, the particle starts at its greatest extent, like pulling a mass on a spring out to a given point a and then letting it go.

At $t = \pi/\omega$, the distribution becomes:

$$|\Psi(x,\pi/\omega)|^2 = \sqrt{\frac{m\omega}{\pi\hbar}} e^{-m\omega(x+a)^2/\hbar} \tag{7}$$

so we have the same shaped curve, but now it is centred at $x = -a$. As time progresses, the wave packet retains its shape, but its position oscillates between $x = a$ and $x = -a$ with a period of $2\pi/\omega$. This is correct for the harmonic oscillator potential $V = m\omega^2 x^2/2$.

Again, using Maple, we can find the mean and mean square position and momentum.

$$\langle x\rangle = \int_{-\infty}^{\infty} |\Psi(x,t)|^2 x dx \tag{8}$$

$$= a\cos(\omega t) \tag{9}$$

which verifies the result from above: the particle's average position oscillates between $x = a$ and $x = -a$ with a period of $2\pi/\omega$.

The mean momentum is found from

$$\langle p\rangle = \frac{\hbar}{i}\int_{-\infty}^{\infty} \Psi^*(x,t)\frac{\partial}{\partial x}\Psi(x,t)dx \tag{10}$$

$$-am\omega\sin(\omega t) \tag{11}$$

Ehrenfest's theoremsays that

$\langle \partial p/\partial t\rangle = -\langle \partial V/\partial x\rangle$ which implies $-ma\omega^2\cos(\omega t) = -m\omega^2\langle x\rangle$. From above we see this is verified.

We can verify that the uncertainty principle is satisfied here. Again using Maple to calculate the integrals, we have

$$\begin{aligned} \left\langle x^2\right\rangle &= \int_{-\infty}^{\infty} |\Psi(x,t)|^2 x^2 dx && (12)\\ &= a^2\cos^2(\omega t) + \frac{\hbar}{2m\omega} && (13) \end{aligned}$$

and

$$\begin{aligned} \left\langle p^2\right\rangle &= -\hbar^2\int_{-\infty}^{\infty} \Psi^*(x,t)\frac{\partial^2}{\partial x^2}\Psi(x,t)dx && (14)\\ &= m^2\omega^2a^2\sin^2(\omega t) + \frac{m\omega\hbar}{2} && (15) \end{aligned}$$

Thus the variances are:

$$\left\langle x^2\right\rangle - \langle x\rangle^2 = \frac{\hbar}{2m\omega} \tag{16}$$

$$\left\langle p^2\right\rangle - \langle p\rangle^2 = \frac{m\omega\hbar}{2} \tag{17}$$

The uncertainty principle thus becomes

$$\sqrt{(\langle x^2\rangle - \langle x\rangle^2)(\langle p^2\rangle - \langle p\rangle^2)} = \hbar/2 \tag{18}$$

Thus this wave function satisfies the uncertainty principle exactly.