DELTA FUNCTION POTENTIAL - MOVING DELTA FUNCTION

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.50.

We've looked at the solution to the Schrödinger equation for a stationary delta function potential well. Here we'll consider the case of a moving delta function well. That is, the potential is

$$V(x) = -\alpha\delta(x - vt) \tag{1}$$

where v is the constant velocity.

A solution of the Schrödinger equation in this case is proposed to be

$$\Psi(x,t) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x-vt|/\hbar^2} e^{-i\left[\left(E+mv^2/2\right)t-mvx\right]/\hbar} \tag{2}$$

To verify this, we consider first the case $x \neq vt$. We need to check that the given wave function satisfies the equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} = i\hbar\frac{\partial\Psi}{\partial t} \tag{3}$$

This can be done by direct substitution. Since the wave function contains an absolute value $|x - vt|$, we can verify the equation for the two cases $x > vt$ and $x < vt$. Using Maple to work out the derivative on the left, we get

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} = -\frac{m^{3/2}\sqrt{\alpha}}{2\hbar^3}\left(\alpha^2 - v^2\hbar^2 \pm 2i\alpha v\hbar\right) e^{-m\alpha|x-vt|/\hbar^2} e^{-i\left[\left(E+mv^2/2\right)t-mvx\right]/\hbar} \tag{4}$$

Working out the right side, we get

$$i\hbar\frac{\partial\Psi}{\partial t} = \frac{\sqrt{m\alpha}}{\hbar^3}\left(\hbar^2\left(E + mv^2/2\right) \mp 2im\alpha v\hbar\right) e^{-m\alpha|x-vt|/\hbar^2} e^{-i\left[\left(E+mv^2/2\right)t-mvx\right]/\hbar} \tag{5}$$

where the top sign in each case is for the region $x > vt$ and the bottom sign for $x < vt$.

If

$$E=-\frac{m\alpha^2}{2\hbar^2} \tag{6}$$

which is the energy of the bound state for the stationary delta function potential, these two expressions are equal, so the wave function as given does satisfy the Schrödinger equation if $x \neq vt$.

To deal with the point $x = vt$, we can work out the *first* derivative of Ψ on either side of this point and compare the two derivatives. We get

$$\frac{\partial\Psi}{\partial x}=\frac{m^{3/2}\sqrt{\alpha}}{\hbar^3}\left(\mp\alpha+i\hbar v\right)e^{-m\alpha|x-vt|/\hbar^2}e^{-i\left[\left(E+mv^2/2\right)t-mvx\right]/\hbar} \tag{7}$$

If we work out these derivatives and then take the limit as $x \to vt$ from each side (substituting for E as above), we get:

$$-\frac{\hbar^2}{2m}\left(\lim_{x\downarrow vt}\frac{\partial\Psi}{\partial x}-\lim_{x\uparrow vt}\frac{\partial\Psi}{\partial x}\right)=\frac{\sqrt{m}\alpha^{3/2}}{\hbar}e^{imt\left(\alpha^2+v^2\hbar^2\right)/2\hbar^3} \tag{8}$$

Thus there is a step function in the first derivative at $x = vt$. The second derivative at this point, using the fact that the derivative of the unit step function is the delta function, is therefore:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}=\frac{\sqrt{m}\alpha^{3/2}}{\hbar}e^{imt\left(\alpha^2+v^2\hbar^2\right)/2\hbar^3}\delta(x-vt)-\frac{\hbar^2}{2m}\Psi_0''(x,t) \tag{9}$$

where $\Psi_0''(x,t)$ is the second derivative of the wave function with respect to x evaluated in the normal way.

Evaluating the wave function itself at $x = vt$ gives:

$$\Psi(vt,t)=\frac{\sqrt{m\alpha}}{\hbar}e^{imt\left(\alpha^2+v^2\hbar^2\right)/2\hbar^3} \tag{10}$$

from which we can see that the first term in 9 is just $\alpha\delta(x-vt)\Psi(vt,t)$. This term is cancelled by the potential $V(x,t)=-\alpha\delta(x-vt)$, so the Schrodinger equation is satisfied at $x = vt$ as well as for other values of x.

To work out the expectation value of the Hamiltonian, we can do it by splitting the integral into two parts. Work out the integrals:

$$\langle H\rangle=-\frac{\hbar^2}{2m}\int_{-\infty}^{vt}\Psi^*_{x<vt}\Psi''_{x<vt}dx-\frac{\hbar^2}{2m}\int_{vt}^{\infty}\Psi^*_{x>vt}\Psi''_{x>vt}dx \tag{11}$$

Using Maple, this comes out to:

$$\langle H \rangle = \frac{m}{4\hbar^2}\left(-\alpha^2 + v^2\hbar^2 - 2i\alpha v\hbar\right) + \frac{m}{4\hbar^2}\left(-\alpha^2 + v^2\hbar^2 + 2i\alpha v\hbar\right) \tag{12}$$

$$= \frac{1}{2}mv^2 - \frac{1}{2}m\frac{\alpha^2}{\hbar^2} \tag{13}$$

The delta function from the potential is cancelled out by the delta function from 9, so the integral involves only the 'ordinary' second derivative of Ψ.

This energy is the original ground state energy $-m\alpha^2/2\hbar^2$ for the delta-function well plus a kinetic energy term $mv^2/2$ for the motion.