REFLECTIONLESS POTENTIAL

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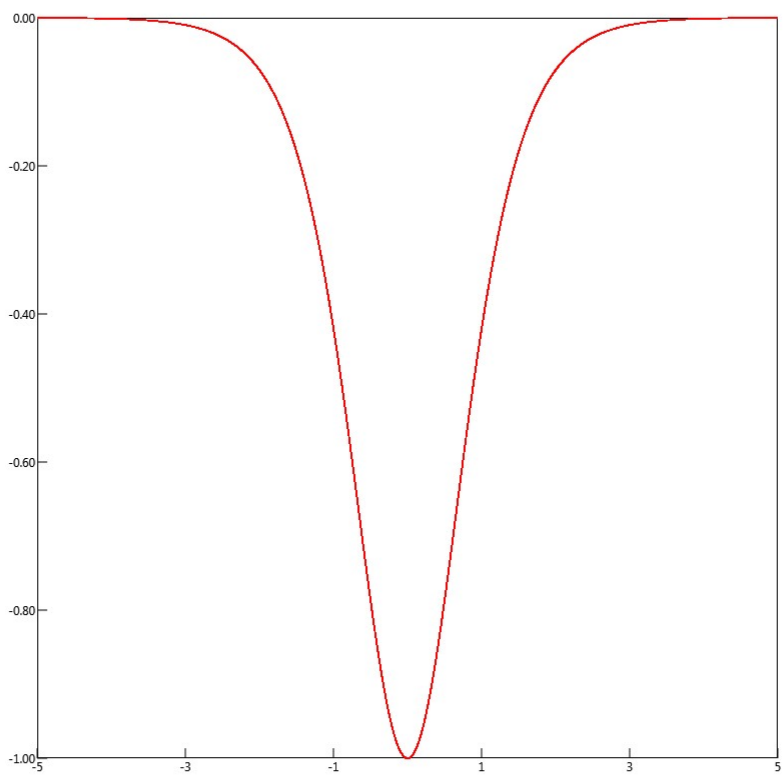
Post date: 27 Aug 2012.

References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.51.

An interesting potential is

$$V(x)=-\frac{\hbar^2a^2}{m}\text{sech}^2(ax) \tag{1}$$

where a is a constant and $\text{sech}(ax)$ is the hyperbolic secant, which is defined as $\text{sech}(ax)\equiv 1/\cosh(ax)$. The general shape of this potential is as shown in the figure.



We can verify by direct substitution that the function

$$\psi_0(x) = A\mathrm{sech}(ax) \tag{2}$$

is a solution. We get

$$\begin{aligned} -\frac{\hbar^2}{2m}\frac{d^2\psi_0}{dx^2} + V\psi_0 &= -\frac{\hbar^2}{2m}Aa^2\mathrm{sech}\,(ax) \qquad (3) \\ &= -\frac{\hbar^2 a^2}{2m}\psi_0 \qquad (4) \end{aligned}$$

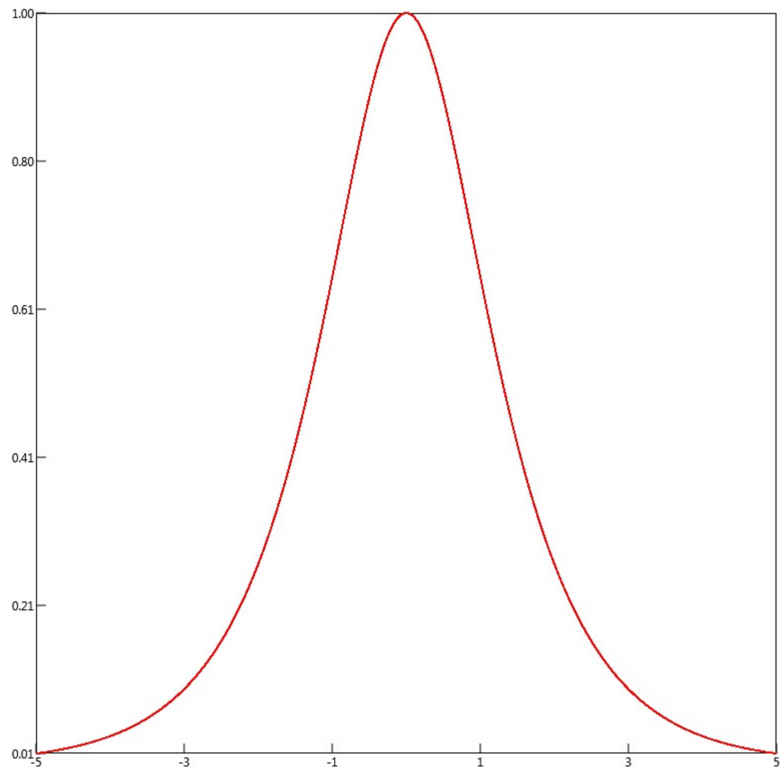
Thus the energy of this state is

$$E_0 = -\frac{\hbar^2 a^2}{2m} \tag{5}$$

We can normalize ψ_0 to find A:

$$\begin{aligned} \int_{-\infty}^{\infty} \psi_0^2 dx &= 1 \qquad (6) \\ A &= \sqrt{a/2} \qquad (7) \end{aligned}$$

A plot of $\psi_0(x)$ looks like this:



For positive energies, we can verify that

$$\psi_k(x) = B\left(\frac{ik - a\tanh(ax)}{ik + a}\right)e^{ikx} \tag{8}$$

is a solution of the Schrodinger equation for any energy by direct substitution. Here, as usual, $k \equiv \sqrt{2mE}/\hbar$.

We get

$$\begin{aligned} -\frac{\hbar^2}{2m}\frac{d^2\psi_k}{dx^2} + V\psi_k &= \frac{\hbar^2 k^2 B}{2m}\left(\frac{ik - a\tanh(ax)}{ik + a}\right) && (9) \\ &= \frac{\hbar^2 k^2}{2m}\psi_k && (10) \\ &= E\psi_k && (11) \end{aligned}$$

The asymptotic behaviour of ψ_k can be found from the limit $\lim_{x\to\infty}\tanh(ax) = 1$. We therefore get:

$$\lim_{x\to\infty}\psi_k(x) = B\frac{ik-a}{ik+a}e^{ikx} \tag{12}$$

$$= -B\frac{(ik-a)^2}{a^2+k^2}e^{ikx} \tag{13}$$

For large negative x $\lim_{x\to-\infty}\tanh(ax) = -1$ so we get

$$\lim_{x\to-\infty}\psi_k(x) = B\frac{ik+a}{ik+a}e^{ikx} \tag{14}$$

$$= Be^{ikx} \tag{15}$$

Thus in both cases, the wave function represents a wave travelling to the right, with no leftward component. That is, there is no reflected wave. The modulus of the wave for large x is

$$\lim_{x\to\infty}|\psi_k(x)|^2 = |B|^2\left|\frac{(ik-a)^2}{a^2+k^2}\right|^2 \tag{16}$$

$$= |B|^2 \tag{17}$$

$$= \lim_{x\to-\infty}|\psi_k(x)|^2 \tag{18}$$

Thus the transmission coefficient is 1 for all positive energies, which means that any particle coming in from the left passes straight through with no reflection. There is, however, a change of phase due to the factor of $\frac{(ik-a)^2}{a^2+k^2}$.

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