

REFLECTIONLESS POTENTIAL

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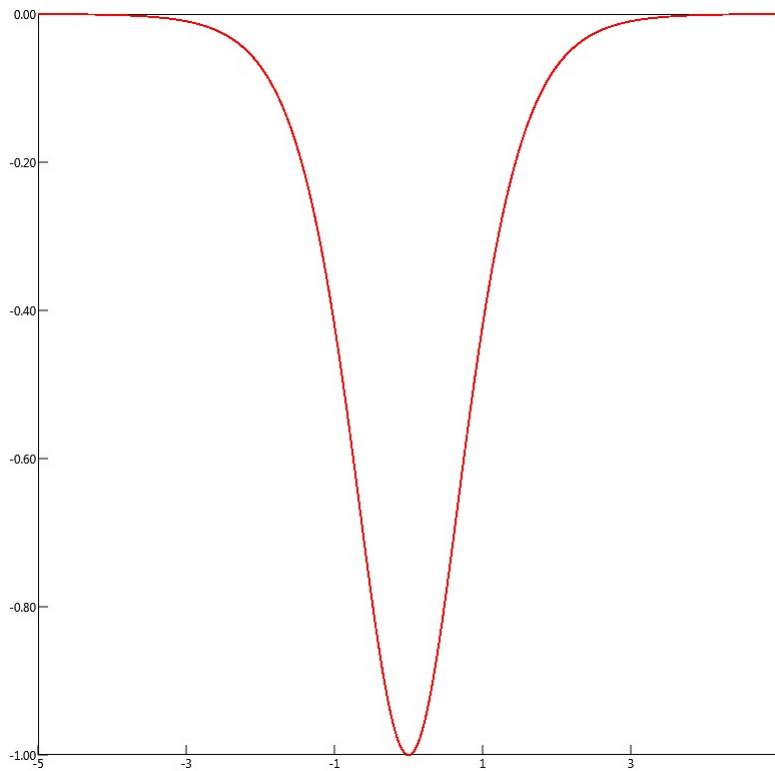
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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.51.

An interesting potential is

$$(1) \quad V(x) = -\frac{\hbar^2 a^2}{m} \operatorname{sech}^2(ax)$$

where a is a constant and $\operatorname{sech}(ax)$ is the hyperbolic secant, which is defined as $\operatorname{sech}(ax) \equiv 1/\cosh(ax)$. The general shape of this potential is as shown in the figure.



We can verify by direct substitution that the function

$$(2) \quad \psi_0(x) = A \operatorname{sech}(ax)$$

is a solution. We get

$$(3) \quad -\frac{\hbar^2}{2m} \frac{d^2 \psi_0}{dx^2} + V \psi_0 = -\frac{\hbar^2}{2m} A a^2 \operatorname{sech}(ax)$$

$$(4) \quad = -\frac{\hbar^2 a^2}{2m} \psi_0$$

Thus the energy of this state is

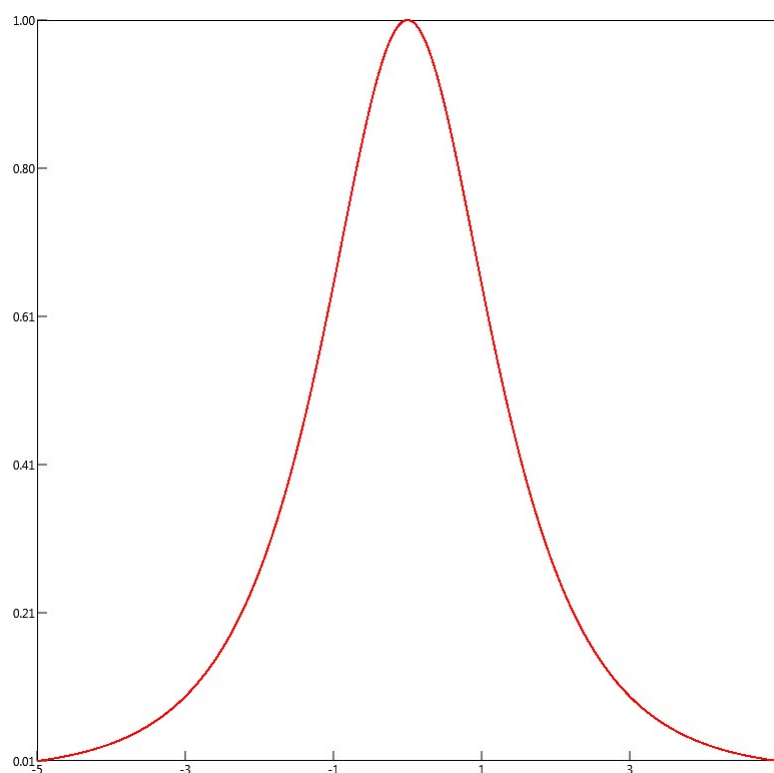
$$(5) \quad E_0 = -\frac{\hbar^2 a^2}{2m}$$

We can normalize ψ_0 to find A :

$$(6) \quad \int_{-\infty}^{\infty} \psi_0^2 dx = 1$$

$$(7) \quad A = \sqrt{a/2}$$

A plot of $\psi_0(x)$ looks like this:



For positive energies, we can verify that

$$(8) \quad \psi_k(x) = B \left(\frac{ik - a \tanh(ax)}{ik + a} \right) e^{ikx}$$

is a solution of the Schrodinger equation for any energy by direct substitution. Here, as usual, $k \equiv \sqrt{2mE}/\hbar$.

We get

$$(9) \quad -\frac{\hbar^2}{2m} \frac{d^2 \psi_k}{dx^2} + V \psi_k = \frac{\hbar^2 k^2 B}{2m} \left(\frac{ik - a \tanh(ax)}{ik + a} \right)$$

$$(10) \quad = \frac{\hbar^2 k^2}{2m} \psi_k$$

$$(11) \quad = E \psi_k$$

The asymptotic behaviour of ψ_k can be found from the limit $\lim_{x \rightarrow \infty} \tanh(ax) = 1$. We therefore get:

$$(12) \quad \lim_{x \rightarrow \infty} \psi_k(x) = B \frac{ik - a}{ik + a} e^{ikx}$$

$$(13) \quad = -B \frac{(ik - a)^2}{a^2 + k^2} e^{ikx}$$

For large negative x $\lim_{x \rightarrow -\infty} \tanh(ax) = -1$ so we get

$$(14) \quad \lim_{x \rightarrow -\infty} \psi_k(x) = B \frac{ik + a}{ik + a} e^{ikx}$$

$$(15) \quad = B e^{ikx}$$

Thus in both cases, the wave function represents a wave travelling to the right, with no leftward component. That is, there is no reflected wave. The modulus of the wave for large x is

$$(16) \quad \lim_{x \rightarrow \infty} |\psi_k(x)|^2 = |B|^2 \left| \frac{(ik - a)^2}{a^2 + k^2} \right|^2$$

$$(17) \quad = |B|^2$$

$$(18) \quad = \lim_{x \rightarrow -\infty} |\psi_k(x)|^2$$

Thus the transmission coefficient is 1 for all positive energies, which means that any particle coming in from the left passes straight through with no reflection. There is, however, a change of phase due to the factor of $\frac{(ik - a)^2}{a^2 + k^2}$.

PINGBACKS

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