An interesting potential is

\[ V(x) = -\frac{\hbar^2 a^2}{m} \text{sech}^2(ax) \]  

(1)

where \( a \) is a constant and \( \text{sech}(ax) \) is the hyperbolic secant, which is defined as \( \text{sech}(ax) \equiv 1/\cosh(ax) \). The general shape of this potential is as shown in the figure.

We can verify by direct substitution that the function
\[ \psi_0(x) = A \text{sech}(ax) \]  

is a solution. We get

\[
-\frac{\hbar^2}{2m} \frac{d^2 \psi_0}{dx^2} + V\psi_0 = -\frac{\hbar^2}{2m} A a^2 \text{sech}(ax)
\]

\[
= -\frac{\hbar^2 a^2}{2m} \psi_0
\]

Thus the energy of this state is

\[ E_0 = -\frac{\hbar^2 a^2}{2m} \]

We can normalize \( \psi_0 \) to find \( A \):

\[
\int_{-\infty}^{\infty} \psi_0^2 dx = 1
\]

\[
A = \sqrt{a/2}
\]

A plot of \( \psi_0(x) \) looks like this:
For positive energies, we can verify that

\[ \psi_k(x) = B \left( \frac{ik - a \tanh(ax)}{ik + a} \right) e^{ikx} \quad (8) \]

is a solution of the Schrödinger equation for any energy by direct substitution. Here, as usual, \( k \equiv \sqrt{2mE/\hbar} \). We get

\[ -\frac{\hbar^2}{2m} \frac{d^2 \psi_k}{dx^2} + V \psi_k = \frac{\hbar^2 k^2 B}{2m} \left( \frac{ik - a \tanh(ax)}{ik + a} \right) \quad (9) \]

\[ = \frac{\hbar^2 k^2}{2m} \psi_k \quad (10) \]

\[ = E \psi_k \quad (11) \]

The asymptotic behaviour of \( \psi_k \) can be found from the limit \( \lim_{x \to \infty} \tanh(ax) = 1 \). We therefore get:
\[
\lim_{x \to \infty} \psi_k(x) = B \frac{ik - a}{ik + a} e^{ikx} \quad (12)
\]
\[
= -B \frac{(ik - a)^2}{a^2 + k^2} e^{ikx} \quad (13)
\]
For large negative \(x\) \(\lim_{x \to -\infty} \tanh(ax) = -1\) so we get
\[
\lim_{x \to -\infty} \psi_k(x) = B \frac{ik + a}{ik + a} e^{ikx} \quad (14)
\]
\[
= Be^{ikx} \quad (15)
\]
Thus in both cases, the wave function represents a wave travelling to the right, with no leftward component. That is, there is no reflected wave. The modulus of the wave for large \(x\) is
\[
\lim_{x \to \infty} |\psi_k(x)|^2 = |B|^2 \frac{(ik - a)^2}{a^2 + k^2} \quad (16)
\]
\[
= |B|^2 \quad (17)
\]
\[
= \lim_{x \to -\infty} |\psi_k(x)|^2 \quad (18)
\]
Thus the transmission coefficient is 1 for all positive energies, which means that any particle coming in from the left passes straight through with no reflection. There is, however, a change of phase due to the factor of \(\frac{(ik - a)^2}{a^2 + k^2}\).

**Pingbacks**

Pingback: [WKB approximation and the reflectionless potential](#)