SCATTERING MATRIX

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.52.

In a system with a localized potential (that is, a potential that is non-zero only for some finite range, such as the delta function or finite square well) we can always analyze the scattering problem, in which particles come in from either right or left (or both) and get transmitted or reflected. In general, the wave function in the left hand region where $V = 0$ is

$$\psi_l = Ae^{ikx} + Be^{-ikx} \tag{1}$$

where

$$k \equiv \frac{\sqrt{2mE}}{\hbar} \tag{2}$$

On the right,

$$\psi_r = Fe^{ikx} + Ge^{-ikx} \tag{3}$$

In the middle, where $V(x) \neq 0$, we can't say what the wave function will be until $V(x)$ is specified. In the cases we've analyzed so far, the only incident particles have been from the left, so we've always taken $G = 0$. However, it's not too difficult to generalize the results we've obtained for the delta function and finite square well to the case where we have incident particles from both directions.

Since a particle coming in from the left will be either transmitted (continue to the right past the potential region) or reflected (travel back to the left), particles incident from the left cannot affect the particle stream travelling *to* the left on the right side of the potential region. By symmetry, particles incident from the right cannot affect the particle stream travelling *to* the right on the left side of the potential. That is, we can always specify A and G in the wave functions above, and express B and F in terms of them. We can write this dependence as a matrix equation

$$\begin{bmatrix} B \\ F \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} A \\ G \end{bmatrix} \tag{4}$$

where the matrix S is called the *scattering matrix*.

For the delta function, with potential $V(x) = -\alpha\delta(x)$ we've seen that in the case where $G = 0$,

$$B = \frac{i\beta}{1-i\beta}A \tag{5}$$

$$F = \frac{1}{1-i\beta}A \tag{6}$$

$$\beta \equiv \frac{m\alpha}{\hbar^2 k} \tag{7}$$

By symmetry, if $A = 0$ so that particles come in only from the right,

$$F = \frac{i\beta}{1-i\beta}G \tag{8}$$

$$B = \frac{1}{1-i\beta}G \tag{9}$$

If both $A \neq 0$ and $G \neq 0$, we can just add up the contributions from the two cases, since they don't interfere with each other, and we get

$$\begin{bmatrix} B \\ F \end{bmatrix} = \frac{1}{1-i\beta}\begin{bmatrix} i\beta & 1 \\ 1 & i\beta \end{bmatrix}\begin{bmatrix} A \\ G \end{bmatrix} \tag{10}$$

For the finite square well of depth V_0, with $G = 0$ we had

$$B = \frac{e^{-2ika}\left(k^2-\mu^2\right)\sin(2\mu a)}{\sin(2\mu a)\left(k^2+\mu^2\right)+2i\mu k\cos(2\mu a)}A \tag{11}$$

$$F = \frac{2i\mu k e^{-2ika}}{\sin(2\mu a)\left(k^2+\mu^2\right)+2i\mu k\cos(2\mu a)}A \tag{12}$$

where

$$\mu = \frac{\sqrt{2m(E+V_0)}}{\hbar} \tag{13}$$

Here, the general scattering matrix is, by symmetry

$$\begin{bmatrix} B \\ F \end{bmatrix} = \frac{e^{-2ika}}{\sin(2\mu a)\left(k^2+\mu^2\right)+2i\mu k\cos(2\mu a)}\begin{bmatrix} \left(k^2-\mu^2\right)\sin(2\mu a) & 2i\mu k \\ 2i\mu k & \left(k^2-\mu^2\right)\sin(2\mu a) \end{bmatrix}\begin{bmatrix} A \\ G \end{bmatrix} \tag{14}$$

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