HERMITIAN OPERATORS - EQUIVALENCE OF CONDITIONS

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 3.3.

We've seen that a Hermitian operator Q satisfies the condition

$$\int_a^b g^* Q f\,dx = \int_a^b f(Qg)^*\,dx \tag{1}$$

where f and g are any functions in Hilbert space that satisfy certain boundary conditions at a and b (see earlier post for a full discussion).

Using the more compact bracket notation, this condition can be written as

$$\langle g|\hat{Q}f\rangle = \langle\hat{Q}g|f\rangle \tag{2}$$

It is also possible to arrive at this result if we start off with the apparently less restrictive condition

$$\langle h|\hat{Q}h\rangle = \langle\hat{Q}h|h\rangle \tag{3}$$

where h is any function in Hilbert space.

To show this, we start with $h = f + g$:

$$\begin{aligned}
\langle h|\hat{Q}h\rangle &= \langle f+g|\hat{Q}f+\hat{Q}g\rangle && (4)\\
&= \langle f|\hat{Q}f\rangle + \langle g|\hat{Q}g\rangle + \langle f|\hat{Q}g\rangle + \langle g|\hat{Q}f\rangle && (5)
\end{aligned}$$

But we are assuming that $\langle h|\hat{Q}h\rangle = \langle\hat{Q}h|h\rangle$ so we can write this equation as:

$$\begin{aligned}
\langle\hat{Q}h|h\rangle &= \langle\hat{Q}f+\hat{Q}g|f+g\rangle && (6)\\
&= \langle\hat{Q}f|f\rangle + \langle\hat{Q}g|g\rangle + \langle\hat{Q}f|g\rangle + \langle\hat{Q}g|f\rangle && (7)
\end{aligned}$$

Using the assumption $\langle h|\hat{Q}h\rangle = \langle\hat{Q}h|h\rangle$ for all functions, we have $\langle f|\hat{Q}f\rangle = \langle\hat{Q}f|f\rangle$ and $\langle g|\hat{Q}g\rangle = \langle\hat{Q}g|g\rangle$ so, equating the two expansions above and cancelling terms, we get

$$\langle f|\hat{Q}g\rangle+\langle g|\hat{Q}f\rangle=\langle\hat{Q}f|g\rangle+\langle\hat{Q}g|f\rangle \tag{8}$$

Repeating this procedure for $h=f+ig$, we have:

$$\begin{aligned}\langle h|\hat{Q}h\rangle &= \langle f+ig|\hat{Q}f+i\hat{Q}g\rangle &(9)\\ &= \langle f|\hat{Q}f\rangle+(-i)(i)\langle g|\hat{Q}g\rangle+i\langle f|\hat{Q}g\rangle-i\langle g|\hat{Q}f\rangle &(10)\\ &= \langle f|\hat{Q}f\rangle+\langle g|\hat{Q}g\rangle+i\langle f|\hat{Q}g\rangle-i\langle g|\hat{Q}f\rangle &(11)\end{aligned}$$

Here, the factor of $-i$ in the second term comes from the complex conjugate of ig.

Finally, we calculate $\langle\hat{Q}h|h\rangle$ for $h=f+ig$ to get

$$\langle\hat{Q}h|h\rangle=\langle\hat{Q}f|f\rangle+\langle\hat{Q}g|g\rangle+i\langle\hat{Q}f|g\rangle-i\langle\hat{Q}g|f\rangle \tag{12}$$

(The $+i$ and $-i$ multiply to give 1 in the second term $\langle\hat{Q}g|g\rangle$.)

Equating the last two expansions, as before, leads to:

$$\langle f|\hat{Q}g\rangle-\langle g|\hat{Q}f\rangle=\langle\hat{Q}f|g\rangle-\langle\hat{Q}g|f\rangle \tag{13}$$

If we now add equations 8 and 13 we get:

$$\langle f|\hat{Q}g\rangle=\langle\hat{Q}f|g\rangle \tag{14}$$

Similarly if we subtract 13 from 8 we get:

$$\langle g|\hat{Q}f\rangle=\langle\hat{Q}g|f\rangle \tag{15}$$

Thus we reclaim the more general Hermitian condition we started with.

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