HERMITIAN OPERATORS - EQUIVALENCE OF CONDITIONS

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Post date: 4 Sep 2012.

References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 3.3.

We’ve seen that a Hermitian operator \( Q \) satisfies the condition

\[
\int_a^b g^* Q f \, dx = \int_a^b f^* (Q g) \, dx
\]  

(1)

where \( f \) and \( g \) are any functions in Hilbert space that satisfy certain boundary conditions at \( a \) and \( b \) (see earlier post for a full discussion).

Using the more compact bracket notation, this condition can be written as

\[
\langle g | \hat{Q} f \rangle = \langle \hat{Q} g | f \rangle
\]

(2)

It is also possible to arrive at this result if we start off with the apparently less restrictive condition

\[
\langle h | \hat{Q} h \rangle = \langle \hat{Q} h | h \rangle
\]

(3)

where \( h \) is any function in Hilbert space.

To show this, we start with \( h = f + g \):

\[
\langle h | \hat{Q} h \rangle = \langle f + g | \hat{Q} f + \hat{Q} g \rangle
\]

(4)

\[
= \langle f | \hat{Q} f \rangle + \langle g | \hat{Q} g \rangle + \langle f | \hat{Q} g \rangle + \langle g | \hat{Q} f \rangle
\]

(5)

But we are assuming that \( \langle h | \hat{Q} h \rangle = \langle \hat{Q} h | h \rangle \) so we can write this equation as:

\[
\langle \hat{Q} h | h \rangle = \langle \hat{Q} f + \hat{Q} g | f + g \rangle
\]

(6)

\[
= \langle \hat{Q} f | f \rangle + \langle \hat{Q} g | g \rangle + \langle \hat{Q} f | g \rangle + \langle \hat{Q} g | f \rangle
\]

(7)

Using the assumption \( \langle h | \hat{Q} h \rangle = \langle \hat{Q} h | h \rangle \) for all functions, we have \( \langle f | \hat{Q} f \rangle = \langle \hat{Q} f | f \rangle \) and \( \langle g | \hat{Q} g \rangle = \langle \hat{Q} g | g \rangle \) so, equating the two expansions above and cancelling terms, we get
\[ \langle f | \hat{Q} g \rangle + \langle g | \hat{Q} f \rangle = \langle \hat{Q} f | g \rangle + \langle \hat{Q} g | f \rangle \]  
(8)

Repeating this procedure for \( h = f + ig \), we have:

\[
\langle h | \hat{Q} h \rangle = \langle f + ig | \hat{Q} f + i \hat{Q} g \rangle
\]
\[= \langle f | \hat{Q} f \rangle + (-i)(i)\langle g | \hat{Q} g \rangle + i(f|\hat{Q}g) - i(g|\hat{Q}f) \]  
(10)
\[
= \langle f | \hat{Q} f \rangle + \langle g | \hat{Q} g \rangle + i\langle f|\hat{Q}g\rangle - i\langle g|\hat{Q}f\rangle \]  
(11)

Here, the factor of \(-i\) in the second term comes from the complex conjugate of \( ig \).

Finally, we calculate \( \langle \hat{Q} h | h \rangle \) for \( h = f + ig \) to get

\[
\langle \hat{Q} h | h \rangle = \langle \hat{Q} f | f \rangle + \langle \hat{Q} g | g \rangle + i\langle \hat{Q} f | g \rangle - i\langle \hat{Q} g | f \rangle \]  
(12)

(The \(+i\) and \(-i\) multiply to give 1 in the second term \( \langle \hat{Q} g | g \rangle \).)

Equating the last two expansions, as before, leads to:

\[
\langle f | \hat{Q} g \rangle - \langle g | \hat{Q} f \rangle = \langle \hat{Q} f | g \rangle - \langle \hat{Q} g | f \rangle \]  
(13)

If we now add equations 8 and 13 we get:

\[
\langle f | \hat{Q} g \rangle = \langle \hat{Q} f | g \rangle \]  
(14)

Similarly if we subtract 13 from 8 we get:

\[
\langle g | \hat{Q} f \rangle = \langle \hat{Q} g | f \rangle \]  
(15)

Thus we reclaim the more general Hermitian condition we started with.

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