

HERMITIAN OPERATORS - EQUIVALENCE OF CONDITIONS

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 3.3.

We've seen that a Hermitian operator Q satisfies the condition

$$(1) \quad \int_a^b g^* Qf \, dx = \int_a^b f(Qg)^* \, dx$$

where f and g are any functions in Hilbert space that satisfy certain boundary conditions at a and b (see earlier post for a full discussion).

Using the more compact bracket notation, this condition can be written as

$$(2) \quad \langle g | \hat{Q}f \rangle = \langle \hat{Q}g | f \rangle$$

It is also possible to arrive at this result if we start off with the apparently less restrictive condition

$$(3) \quad \langle h | \hat{Q}h \rangle = \langle \hat{Q}h | h \rangle$$

where h is any function in Hilbert space.

To show this, we start with $h = f + g$:

$$(4) \quad \langle h | \hat{Q}h \rangle = \langle f + g | \hat{Q}f + \hat{Q}g \rangle$$

$$(5) \quad = \langle f | \hat{Q}f \rangle + \langle g | \hat{Q}g \rangle + \langle f | \hat{Q}g \rangle + \langle g | \hat{Q}f \rangle$$

But we are assuming that $\langle h | \hat{Q}h \rangle = \langle \hat{Q}h | h \rangle$ so we can write this equation as:

$$(6) \quad \langle \hat{Q}h | h \rangle = \langle \hat{Q}f + \hat{Q}g | f + g \rangle$$

$$(7) \quad = \langle \hat{Q}f | f \rangle + \langle \hat{Q}g | g \rangle + \langle \hat{Q}f | g \rangle + \langle \hat{Q}g | f \rangle$$

Using the assumption $\langle h | \hat{Q}h \rangle = \langle \hat{Q}h | h \rangle$ for all functions, we have $\langle f | \hat{Q}f \rangle = \langle \hat{Q}f | f \rangle$ and $\langle g | \hat{Q}g \rangle = \langle \hat{Q}g | g \rangle$ so, equating the two expansions above and cancelling terms, we get

$$(8) \quad \langle f|\hat{Q}g\rangle + \langle g|\hat{Q}f\rangle = \langle \hat{Q}f|g\rangle + \langle \hat{Q}g|f\rangle$$

Repeating this procedure for $h = f + ig$, we have:

$$(9) \quad \langle h|\hat{Q}h\rangle = \langle f + ig|\hat{Q}f + i\hat{Q}g\rangle$$

$$(10) \quad = \langle f|\hat{Q}f\rangle + (-i)(i)\langle g|\hat{Q}g\rangle + i\langle f|\hat{Q}g\rangle - i\langle g|\hat{Q}f\rangle$$

$$(11) \quad = \langle f|\hat{Q}f\rangle + \langle g|\hat{Q}g\rangle + i\langle f|\hat{Q}g\rangle - i\langle g|\hat{Q}f\rangle$$

Here, the factor of $-i$ in the second term comes from the complex conjugate of ig .

Finally, we calculate $\langle \hat{Q}h|h\rangle$ for $h = f + ig$ to get

$$(12) \quad \langle \hat{Q}h|h\rangle = \langle \hat{Q}f|f\rangle + \langle \hat{Q}g|g\rangle + i\langle \hat{Q}f|g\rangle - i\langle \hat{Q}g|f\rangle$$

(The $+i$ and $-i$ multiply to give 1 in the second term $\langle \hat{Q}g|g\rangle$.)

Equating the last two expansions, as before, leads to:

$$(13) \quad \langle f|\hat{Q}g\rangle - \langle g|\hat{Q}f\rangle = \langle \hat{Q}f|g\rangle - \langle \hat{Q}g|f\rangle$$

If we now add equations 8 and 13 we get:

$$(14) \quad \langle f|\hat{Q}g\rangle = \langle \hat{Q}f|g\rangle$$

Similarly if we subtract 13 from 8 we get:

$$(15) \quad \langle g|\hat{Q}f\rangle = \langle \hat{Q}g|f\rangle$$

Thus we reclaim the more general Hermitian condition we started with.

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