

## HERMITIAN OPERATORS - EQUIVALENCE OF CONDITIONS

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 3.3.

We've seen that a Hermitian operator  $Q$  satisfies the condition

$$\int_a^b g^* Qf dx = \int_a^b f(Qg)^* dx \quad (1)$$

where  $f$  and  $g$  are any functions in Hilbert space that satisfy certain boundary conditions at  $a$  and  $b$  (see earlier post for a full discussion).

Using the more compact bracket notation, this condition can be written as

$$\langle g|\hat{Q}f\rangle = \langle \hat{Q}g|f\rangle \quad (2)$$

It is also possible to arrive at this result if we start off with the apparently less restrictive condition

$$\langle h|\hat{Q}h\rangle = \langle \hat{Q}h|h\rangle \quad (3)$$

where  $h$  is any function in Hilbert space.

To show this, we start with  $h = f + g$ :

$$\langle h|\hat{Q}h\rangle = \langle f + g|\hat{Q}f + \hat{Q}g\rangle \quad (4)$$

$$= \langle f|\hat{Q}f\rangle + \langle g|\hat{Q}g\rangle + \langle f|\hat{Q}g\rangle + \langle g|\hat{Q}f\rangle \quad (5)$$

But we are assuming that  $\langle h|\hat{Q}h\rangle = \langle \hat{Q}h|h\rangle$  so we can write this equation as:

$$\langle \hat{Q}h|h\rangle = \langle \hat{Q}f + \hat{Q}g|f + g\rangle \quad (6)$$

$$= \langle \hat{Q}f|f\rangle + \langle \hat{Q}g|g\rangle + \langle \hat{Q}f|g\rangle + \langle \hat{Q}g|f\rangle \quad (7)$$

Using the assumption  $\langle h|\hat{Q}h\rangle = \langle \hat{Q}h|h\rangle$  for all functions, we have  $\langle f|\hat{Q}f\rangle = \langle \hat{Q}f|f\rangle$  and  $\langle g|\hat{Q}g\rangle = \langle \hat{Q}g|g\rangle$  so, equating the two expansions above and cancelling terms, we get

$$\langle f|\hat{Q}g\rangle + \langle g|\hat{Q}f\rangle = \langle \hat{Q}f|g\rangle + \langle \hat{Q}g|f\rangle \quad (8)$$

Repeating this procedure for  $h = f + ig$ , we have:

$$\langle h|\hat{Q}h\rangle = \langle f + ig|\hat{Q}f + i\hat{Q}g\rangle \quad (9)$$

$$= \langle f|\hat{Q}f\rangle + (-i)(i)\langle g|\hat{Q}g\rangle + i\langle f|\hat{Q}g\rangle - i\langle g|\hat{Q}f\rangle \quad (10)$$

$$= \langle f|\hat{Q}f\rangle + \langle g|\hat{Q}g\rangle + i\langle f|\hat{Q}g\rangle - i\langle g|\hat{Q}f\rangle \quad (11)$$

Here, the factor of  $-i$  in the second term comes from the complex conjugate of  $ig$ .

Finally, we calculate  $\langle \hat{Q}h|h\rangle$  for  $h = f + ig$  to get

$$\langle \hat{Q}h|h\rangle = \langle \hat{Q}f|f\rangle + \langle \hat{Q}g|g\rangle + i\langle \hat{Q}f|g\rangle - i\langle \hat{Q}g|f\rangle \quad (12)$$

(The  $+i$  and  $-i$  multiply to give 1 in the second term  $\langle \hat{Q}g|g\rangle$ .)

Equating the last two expansions, as before, leads to:

$$\langle f|\hat{Q}g\rangle - \langle g|\hat{Q}f\rangle = \langle \hat{Q}f|g\rangle - \langle \hat{Q}g|f\rangle \quad (13)$$

If we now add equations 8 and 13 we get:

$$\langle f|\hat{Q}g\rangle = \langle \hat{Q}f|g\rangle \quad (14)$$

Similarly if we subtract 13 from 8 we get:

$$\langle g|\hat{Q}f\rangle = \langle \hat{Q}g|f\rangle \quad (15)$$

Thus we reclaim the more general Hermitian condition we started with.

#### PINGBACKS

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