HERMITIAN OPERATORS - A FEW THEOREMS

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We’ll look at a few simple theorems about hermitian operators.

(a) The sum of two hermitian operators is also hermitian. From the definition:

\[
\langle f | (\hat{Q} + \hat{R}) f \rangle = \langle f | \hat{Q} f \rangle + \langle f | \hat{R} f \rangle = \langle \hat{Q} f | f \rangle + \langle \hat{R} f | f \rangle = \langle (\hat{Q} + \hat{R}) f | f \rangle
\]

(b) If \(\hat{Q}\) is hermitian, then \(\alpha \hat{Q}\) is hermitian if \(\alpha\) is real. First we note:

\[
\langle f | \alpha \hat{Q} f \rangle = \alpha \langle f | \hat{Q} f \rangle = \alpha \langle \hat{Q} f | f \rangle
\]

Next, for \(\alpha \hat{Q}\) to be hermitian:

\[
\langle \alpha \hat{Q} f | f \rangle = \alpha^* \langle \hat{Q} f | f \rangle = \alpha \langle \hat{Q} f | f \rangle
\]

so we must have \(\alpha = \alpha^*\).

(c) The product of two hermitian operators \(\hat{Q}\) and \(\hat{R}\) is hermitian if the operators commute. We have:

\[
\langle f | \hat{Q}(\hat{R} f) \rangle = \langle \hat{Q} f | \hat{R} f \rangle = \langle \hat{R}(\hat{Q} f) | f \rangle
\]

In order for this to be hermitian, we must have

\[
\langle \hat{R}(\hat{Q} f) | f \rangle = \langle \hat{Q}(\hat{R} f) | f \rangle
\]

so the operators must commute.
(d) The position operator $\hat{x}$ is hermitian since it is real and its operation is simply multiplication, so it doesn’t matter where it appears in the inner product:

$$\langle f | \hat{x} f \rangle = \langle \hat{x} f | f \rangle$$  \hspace{1cm} (10)

To test the hamiltonian operator $H = \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x)$, we note that the potential part $V(x)$ is a real, multiplicative operator so is hermitian by the same reasoning as for $\hat{x}$. The kinetic part consists of a real constant $-\hbar^2/2m$ multiplied by a second derivative, so by part (b) we need show only that the second derivative is hermitian. We can do this by introducing a test function $f$ for the derivative to operate on, and then integrating by parts twice:

$$\langle f | f'' \rangle = \int f^* f'' dx$$  \hspace{1cm} (11)

$$= f^* f' - \int (f^*)' f' dx$$  \hspace{1cm} (12)

$$= f^* f' - (f^*)' f + \int (f^*)'' f' dx$$  \hspace{1cm} (13)

$$= f^* f' - (f^*)' f + \langle f'' | f \rangle$$  \hspace{1cm} (14)

We can throw away the two terms $f^* f' - f'^* f$ under the usual assumption that all physical functions tend to zero at the boundaries, so we are left with

$$\langle f | f'' \rangle = \langle f'' | f \rangle$$  \hspace{1cm} (15)

The hamiltonian is thus the sum of two hermitian operators, so is itself hermitian by part (a).

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