

## HERMITIAN OPERATORS - A FEW THEOREMS

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 3.4.

We'll look at a few simple theorems about hermitian operators.

(a) The sum of two hermitian operators is also hermitian. From the definition:

$$(0.1) \quad \langle f | (\hat{Q} + \hat{R}) f \rangle = \langle f | \hat{Q} f \rangle + \langle f | \hat{R} f \rangle$$

$$(0.2) \quad = \langle \hat{Q} f | f \rangle + \langle \hat{R} f | f \rangle$$

$$(0.3) \quad = \langle (\hat{Q} + \hat{R}) f | f \rangle$$

(b) If  $\hat{Q}$  is hermitian, then  $\alpha\hat{Q}$  is hermitian if  $\alpha$  is real. First we note:

$$(0.4) \quad \langle f | \alpha \hat{Q} f \rangle = \alpha \langle f | \hat{Q} f \rangle = \alpha \langle \hat{Q} f | f \rangle$$

Next, for  $\alpha\hat{Q}$  to be hermitian:

$$(0.5) \quad \langle \alpha \hat{Q} f | f \rangle = \alpha^* \langle \hat{Q} f | f \rangle$$

$$(0.6) \quad = \alpha \langle \hat{Q} f | f \rangle$$

so we must have  $\alpha = \alpha^*$ .

(c) The product of two hermitian operators  $\hat{Q}$  and  $\hat{R}$  is hermitian if the operators commute. We have:

$$(0.7) \quad \langle f | \hat{Q}(\hat{R}f) \rangle = \langle \hat{Q}f | \hat{R}f \rangle$$

$$(0.8) \quad = \langle \hat{R}(\hat{Q}f) | f \rangle$$

In order for this to be hermitian, we must have

$$(0.9) \quad \langle \hat{R}(\hat{Q}f) | f \rangle = \langle \hat{Q}(\hat{R}f) | f \rangle$$

so the operators must commute.

(d) The position operator  $\hat{x}$  is hermitian since it is real and its operation is simply multiplication, so it doesn't matter where it appears in the inner product:

$$(0.10) \quad \langle f|\hat{x}f\rangle = \langle \hat{x}f|f\rangle$$

To test the hamiltonian operator  $H = \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x)$ , we note that the potential part  $V(x)$  is a real, multiplicative operator so is hermitian by the same reasoning as for  $\hat{x}$ . The kinetic part consists of a real constant  $-\hbar^2/2m$  multiplied by a second derivative, so by part (b) we need show only that the second derivative is hermitian. We can do this by introducing a test function  $f$  for the derivative to operate on, and then integrating by parts twice:

$$(0.11) \quad \langle f|f''\rangle = \int f^* f'' dx$$

$$(0.12) \quad = f^* f' - \int (f^*)' f' dx$$

$$(0.13) \quad = f^* f' - (f^*)' f + \int (f^*)'' f dx$$

$$(0.14) \quad = f^* f' - (f^*)' f + \langle f''|f\rangle$$

We can throw away the two terms  $f^* f' - (f^*)' f$  under the usual assumption that all physical functions tend to zero at the boundaries, so we are left with

$$(0.15) \quad \langle f|f''\rangle = \langle f''|f\rangle$$

The hamiltonian is thus the sum of two hermitian operators, so is itself hermitian by part (a).

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