

HERMITIAN OPERATORS - A FEW THEOREMS

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 3.4.

We'll look at a few simple theorems about hermitian operators.

(a) The sum of two hermitian operators is also hermitian. From the definition:

$$\langle f | (\hat{Q} + \hat{R})f \rangle = \langle f | \hat{Q}f \rangle + \langle f | \hat{R}f \rangle \quad (1)$$

$$= \langle \hat{Q}f | f \rangle + \langle \hat{R}f | f \rangle \quad (2)$$

$$= \langle (\hat{Q} + \hat{R})f | f \rangle \quad (3)$$

(b) If \hat{Q} is hermitian, then $\alpha\hat{Q}$ is hermitian if α is real. First we note:

$$\langle f | \alpha\hat{Q}f \rangle = \alpha \langle f | \hat{Q}f \rangle = \alpha \langle \hat{Q}f | f \rangle \quad (4)$$

Next, for $\alpha\hat{Q}$ to be hermitian:

$$\langle \alpha\hat{Q}f | f \rangle = \alpha^* \langle \hat{Q}f | f \rangle \quad (5)$$

$$= \alpha \langle \hat{Q}f | f \rangle \quad (6)$$

so we must have $\alpha = \alpha^*$.

(c) The product of two hermitian operators \hat{Q} and \hat{R} is hermitian if the operators commute. We have:

$$\langle f | \hat{Q}(\hat{R}f) \rangle = \langle \hat{Q}f | \hat{R}f \rangle \quad (7)$$

$$= \langle \hat{R}(\hat{Q}f) | f \rangle \quad (8)$$

In order for this to be hermitian, we must have

$$\langle \hat{R}(\hat{Q}f) | f \rangle = \langle \hat{Q}(\hat{R}f) | f \rangle \quad (9)$$

so the operators must commute.

(d) The position operator \hat{x} is hermitian since it is real and its operation is simply multiplication, so it doesn't matter where it appears in the inner product:

$$\langle f|\hat{x}f\rangle = \langle \hat{x}f|f\rangle \quad (10)$$

To test the hamiltonian operator $H = \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x)$, we note that the potential part $V(x)$ is a real, multiplicative operator so is hermitian by the same reasoning as for \hat{x} . The kinetic part consists of a real constant $-\hbar^2/2m$ multiplied by a second derivative, so by part (b) we need show only that the second derivative is hermitian. We can do this by introducing a test function f for the derivative to operate on, and then integrating by parts twice:

$$\langle f|f''\rangle = \int f^* f'' dx \quad (11)$$

$$= f^* f' - \int (f^*)' f' dx \quad (12)$$

$$= f^* f' - (f^*)' f + \int (f^*)'' f dx \quad (13)$$

$$= f^* f' - (f^*)' f + \langle f''|f\rangle \quad (14)$$

We can throw away the two terms $f^* f' - f^{*'} f$ under the usual assumption that all physical functions tend to zero at the boundaries, so we are left with

$$\langle f|f''\rangle = \langle f''|f\rangle \quad (15)$$

The hamiltonian is thus the sum of two hermitian operators, so is itself hermitian by part (a).

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