

HERMITIAN CONJUGATE OF AN OPERATOR

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 3.5.

We've had a look at some properties of hermitian operators in the last few posts. Here we'll look at the *hermitian conjugate* or *adjoint* of an operator.

The adjoint of an operator \hat{Q} is defined as the operator \hat{Q}^\dagger such that

$$(1) \quad \langle f | \hat{Q}g \rangle = \langle \hat{Q}^\dagger f | g \rangle$$

For a hermitian operator, we must have

$$(2) \quad \langle f | \hat{Q}g \rangle = \langle \hat{Q}f | g \rangle$$

which means a hermitian operator is equal to its own adjoint.

We can find the adjoints of some operators we've already met.

(1) The position operator x : Since x is hermitian, its adjoint is also x .

(2) The imaginary number i : We must have $\langle f | ig \rangle = \langle \hat{Q}^\dagger f | g \rangle$ so $\hat{Q}^\dagger = -i$.

(3) For the operator d/dx , we can use integration by parts to find:

$$(3) \quad \left\langle f \left| \frac{d}{dx} g \right. \right\rangle = \int f^* g' dx$$

$$(4) \quad = f^* g - \int f'^* g dx$$

$$(5) \quad = - \left\langle \frac{d}{dx} f \left| g \right. \right\rangle$$

(where we throw away the integrated terms under the usual assumption that they are zero at the limits of integration) so $\hat{Q}^\dagger = -\frac{d}{dx}$.

If we have the product of two operators, we find that

$$(6) \quad \langle f | \hat{Q}(\hat{R}f) \rangle = \langle \hat{Q}^\dagger f | \hat{R}f \rangle$$

$$(7) \quad = \langle \hat{R}^\dagger (\hat{Q}^\dagger f) | f \rangle$$

Thus

$$(8) \quad (\hat{Q}\hat{R})^\dagger = \hat{R}^\dagger \hat{Q}^\dagger$$

Another interesting case is the harmonic oscillator raising operator, which is

$$(9) \quad a_+ = \frac{1}{\sqrt{2\hbar m\omega}}(-ip + m\omega x)$$

Since p and x are hermitian, and every other term apart from i is a real constant, we can use the results above to see that:

$$(10) \quad a_+^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}(ip + m\omega x) = a_-$$

Thus the raising and lowering operators are hermitian conjugates of each other.