

## HERMITIAN CONJUGATE OF AN OPERATOR

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Post date: 5 Sep 2012.

References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 3.5.

We've had a look at some properties of hermitian operators in the last few posts. Here we'll look at the *hermitian conjugate* or *adjoint* of an operator.

The adjoint of an operator  $\hat{Q}$  is defined as the operator  $\hat{Q}^\dagger$  such that

$$\langle f | \hat{Q}g \rangle = \langle \hat{Q}^\dagger f | g \rangle \quad (1)$$

For a hermitian operator, we must have

$$\langle f | \hat{Q}g \rangle = \langle \hat{Q}f | g \rangle \quad (2)$$

which means a hermitian operator is equal to its own adjoint.

We can find the adjoints of some operators we've already met.

(1) The position operator  $x$ : Since  $x$  is hermitian, its adjoint is also  $x$ .

(2) The imaginary number  $i$ : We must have  $\langle f | ig \rangle = \langle \hat{Q}^\dagger f | g \rangle$  so  $\hat{Q}^\dagger = -i$ .

(3) For the operator  $d/dx$ , we can use integration by parts to find:

$$\left\langle f \left| \frac{d}{dx} g \right. \right\rangle = \int f^* g' dx \quad (3)$$

$$= f^* g - \int f'^* g dx \quad (4)$$

$$= - \left\langle \frac{d}{dx} f \left| g \right. \right\rangle \quad (5)$$

(where we throw away the integrated terms under the usual assumption that they are zero at the limits of integration) so  $\hat{Q}^\dagger = -\frac{d}{dx}$ .

If we have the product of two operators, we find that

$$\langle f | \hat{Q}(\hat{R}f) \rangle = \langle \hat{Q}^\dagger f | \hat{R}f \rangle \quad (6)$$

$$= \langle \hat{R}^\dagger (\hat{Q}^\dagger f) | f \rangle \quad (7)$$

Thus

$$(\hat{Q}\hat{R})^\dagger = \hat{R}^\dagger\hat{Q}^\dagger \quad (8)$$

Another interesting case is the harmonic oscillator raising operator, which is

$$a_+ = \frac{1}{\sqrt{2\hbar m\omega}}(-ip + m\omega x) \quad (9)$$

Since  $p$  and  $x$  are hermitian, and every other term apart from  $i$  is a real constant, we can use the results above to see that:

$$a_+^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}(ip + m\omega x) = a_- \quad (10)$$

Thus the raising and lowering operators are hermitian conjugates of each other.