HERMITIAN CONJUGATE OF AN OPERATOR

We’ve had a look at some properties of hermitian operators in the last few posts. Here we’ll look at the hermitian conjugate or adjoint of an operator.

The adjoint of an operator $\hat{Q}$ is defined as the operator $\hat{Q}^\dagger$ such that

$$\langle f | \hat{Q} g \rangle = \langle \hat{Q}^\dagger f | g \rangle$$

(1)

For a hermitian operator, we must have

$$\langle f | \hat{Q} g \rangle = \langle \hat{Q} f | g \rangle$$

(2)

which means a hermitian operator is equal to its own adjoint.

We can find the adjoints of some operators we’ve already met.

1. The position operator $x$: Since $x$ is hermitian, its adjoint is also $x$.
2. The imaginary number $i$: We must have $\langle f | ig \rangle = \langle \hat{Q}^\dagger f | g \rangle$ so $\hat{Q}^\dagger = -i$.
3. For the operator $d/dx$, we can use integration by parts to find:

$$\langle f \left| \frac{d}{dx} g \right\rangle = \int f^* g' dx$$

(3)

$$= f^* g - \int f^* g' dx$$

(4)

$$= -\langle \frac{d}{dx} f | g \rangle$$

(5)

(where we throw away the integrated terms under the usual assumption that they are zero at the limits of integration) so $\hat{Q}^\dagger = -\frac{d}{dx}$.

If we have the product of two operators, we find that

$$\langle f | \hat{Q} (\hat{R} f) \rangle = \langle \hat{Q}^\dagger f | \hat{R} f \rangle$$

(6)

$$= \langle \hat{R}^\dagger (\hat{Q}^\dagger f) | f \rangle$$

(7)
Thus

\((\hat{Q}\hat{R})^\dagger = \hat{R}^\dagger\hat{Q}^\dagger\)  \hspace{2cm} (8)

Another interesting case is the harmonic oscillator raising operator, which is

\[ a_+ = \frac{1}{\sqrt{2\hbar m\omega}}(-ip + m\omega x) \]  \hspace{2cm} (9)

Since \(p\) and \(x\) are hermitian, and every other term apart from \(i\) is a real constant, we can use the results above to see that:

\[ a_+\dagger = \frac{1}{\sqrt{2\hbar m\omega}}(ip + m\omega x) = a_- \]  \hspace{2cm} (10)

Thus the raising and lowering operators are hermitian conjugates of each other.