# HERMITIAN CONJUGATE OF AN OPERATOR

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 3.5.

We've had a look at some properties of hermitian operators in the last few posts. Here we'll look at the *hermitian conjugate* or *adjoint* of an operator.

The adjoint of an operator $\hat{Q}$ is defined as the operator $\hat{Q}^{\dagger}$ such that

$$\left\langle f\right|\left.\hat{Q}g\right\rangle =\left\langle \hat{Q}^{\dagger}f\right|\left.g\right\rangle \tag{1}$$

For a hermitian operator, we must have

$$\left\langle f\right|\left.\hat{Q}g\right\rangle =\left\langle \hat{Q}f\right|\left.g\right\rangle \tag{2}$$

which means a hermitian operator is equal to its own adjoint.

We can find the adjoints of some operators we've already met.

(1) The position operator $x$: Since $x$ is hermitian, its adjoint is also $x$.
(2) The imaginary number $i$: We must have $\langle f|ig\rangle=\left\langle \hat{Q}^{\dagger}f|g\right\rangle$ so $\hat{Q}^{\dagger}=-i$.
(3) For the operator $d/dx$, we can use integration by parts to find:

$$\begin{aligned}
\left\langle f\left|\frac{d}{dx}g\right.\right\rangle &= \int f^{*}g^{\prime}dx \qquad (3)\\
&= f^{*}g-\int f^{*\prime}gdx \qquad (4)\\
&= -\left\langle \frac{d}{dx}f\right|\left.g\right\rangle \qquad (5)
\end{aligned}$$

(where we throw away the integrated terms under the usual assumption that they are zero at the limits of integration) so $\hat{Q}^{\dagger}=-\frac{d}{dx}$.

If we have the product of two operators, we find that

$$\begin{aligned}
\left\langle f|\hat{Q}(\hat{R}f)\right\rangle &= \left\langle \hat{Q}^{\dagger}f|\hat{R}f\right\rangle \qquad (6)\\
&= \left\langle \hat{R}^{\dagger}(\hat{Q}^{\dagger}f)|f\right\rangle \qquad (7)
\end{aligned}$$

Thus

$$\left(\hat{Q}\hat{R}\right)^{\dagger}=\hat{R}^{\dagger}\hat{Q}^{\dagger} \tag{8}$$

Another interesting case is the harmonic oscillator raising operator, which is

$$a_{+}=\frac{1}{\sqrt{2\hbar m\omega}}\left(-ip+m\omega x\right) \tag{9}$$

Since $p$ and $x$ are hermitian, and every other term apart from $i$ is a real constant, we can use the results above to see that:

$$a_{+}^{\dagger}=\frac{1}{\sqrt{2\hbar m\omega}}\left(ip+m\omega x\right)=a_{-} \tag{10}$$

Thus the raising and lowering operators are hermitian conjugates of each other.