

MOMENTUM: EIGENVALUES AND NORMALIZATION

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Section 3.3.2; Problem 3.9.

The example of a periodic function which we studied earlier had discrete eigenvalues for both the first and second derivative of the periodic variable. In particular, for the operator $id/d\phi$ we found that the eigenvalues are all integers, with eigenfunctions $e^{in\phi}$ since

$$(0.1) \quad i \frac{d}{d\phi} e^{in\phi} = -ne^{in\phi}$$

This operator bears a strong resemblance to the momentum operator in one dimension, which is $\hat{p} = -i\hbar d/dx$. However, if we try to find the eigenvalues and eigenfunctions of \hat{p} , we run into a bit of a problem. We try to solve, for some eigenvalue p :

$$(0.2) \quad \hat{p}f = pf$$

$$(0.3) \quad -i\hbar \frac{d}{dx} f = pf$$

This has the solution

$$(0.4) \quad f_p(x) = Ae^{ipx/\hbar}$$

for some constant A . Ordinarily, at this stage, we would impose some boundary condition on the solution to obtain acceptable values of p . The problem is that we'd like to define this function over all x and, if we try to do this, the function is not normalizable for any value of p . At first glance, we might think that if we chose p to be purely imaginary as in $p = \alpha i$, it might work since we get

$$(0.5) \quad f(x) = Ae^{-\alpha x/\hbar}$$

but of course this tends to infinity at large negative x so that doesn't work.

In fact if p has a non-zero imaginary part, $f(x)$ goes to infinity at one end of its domain. So we're restricted to looking at real values of p .

In that case, $f(x)$ is periodic and thus is still not normalizable. Thus there are no eigenfunctions of the momentum operator that lie in Hilbert space (which, remember, is the vector space of square-integrable functions).

What happens if do the normalization integral anyway? That is, we try

$$(0.6) \quad \int_{-\infty}^{\infty} f_{p_1}^*(x) f_{p_2}(x) dx = |A|^2 \int_{-\infty}^{\infty} e^{i(p_2-p_1)x/\hbar} dx$$

By using the variable transformation $\xi \equiv x/\hbar$, we get

$$(0.7) \quad \int_{-\infty}^{\infty} f_{p_1}^*(x) f_{p_2}(x) dx = |A|^2 \hbar \int_{-\infty}^{\infty} e^{i(p_2-p_1)\xi} d\xi$$

It's at this point that we invoke the dodgy formula involving the Dirac delta function that we obtained a while back. Using this, we can write the integral as a delta function, and we get

$$(0.8) \quad \int_{-\infty}^{\infty} f_{p_1}^*(x) f_{p_2}(x) dx = 2\pi |A|^2 \hbar \delta(p_2 - p_1)$$

This is sort of like a normalization condition, in that the integral is zero when $p_1 \neq p_2$ (that is, if you believe that the integral really does evaluate to a delta function), and non-zero (infinite, in fact) if $p_1 = p_2$. In fact, if we take the constant A to be

$$(0.9) \quad A = \frac{1}{\sqrt{2\pi\hbar}}$$

and use the bra-ket notation for the integral, we can write

$$(0.10) \quad \langle f_{p_1} | f_{p_2} \rangle = \delta(p_2 - p_1)$$

We can also express an arbitrary function $g(x)$ as a Fourier transform over p by writing

$$(0.11) \quad g(x) = \int_{-\infty}^{\infty} c(p) f_p(x) dp$$

$$(0.12) \quad = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} c(p) e^{ipx/\hbar} dp$$

$$(0.13) \quad g(\hbar\xi) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} c(p) e^{ip\xi} dp$$

From Plancherel's theorem, we can invert this relation to get $c(p)$:

$$(0.14) \quad c(p) = \sqrt{\frac{\hbar}{2\pi}} \int_{-\infty}^{\infty} g(\hbar\xi) e^{-ip\xi} d\xi$$

$$(0.15) \quad = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} g(x) e^{-ipx/\hbar} dx$$

$$(0.16) \quad = \langle f_p | g \rangle$$

In general, hermitian operators with continuous eigenvalues don't have normalizable eigenfunctions and have to be analyzed in this way. In particular, the hamiltonian (energy) of a system can have an entirely discrete spectrum (infinite square well or harmonic oscillator), a totally continuous spectrum (free particle, delta function barrier or finite square barrier) or a mixture of the two (delta function well or finite square well).

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