MOMENTUM SPACE: MEAN POSITION

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 3.12.

The momentum space wave function is the Fourier transform of the regular position space wave function:

$$\Phi(p,t)=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\Psi(x,t)e^{-ipx/\hbar}dx \tag{1}$$

In momentum space, the mean value of the momentum is simply

$$\begin{aligned}\langle p\rangle &= \langle\Phi|\,p\Phi\rangle &(2)\\ &= \int_{-\infty}^{\infty}\Phi^{*}p\Phi dp &(3)\end{aligned}$$

The mean position turns out to be

$$\langle x\rangle=\int\Phi^{*}\left(-\frac{\hbar}{i}\frac{\partial}{\partial p}\right)\Phi dp \tag{4}$$

We can show this as follows. Starting with the definition above, we get

$$-\frac{\hbar}{i}\frac{\partial}{\partial p}\Phi=\frac{1}{\sqrt{2\pi\hbar}}\int xe^{-ipx/\hbar}\Psi dx \tag{5}$$

Substituting this together with the expression for Φ^{*} from above into the RHS of 4 we get:

$$\int\Phi^{*}\left(-\frac{\hbar}{i}\frac{\partial}{\partial p}\right)\Phi dp=\frac{1}{2\pi\hbar}\int\left(\int e^{ipx'/\hbar}\Psi^{*}(x')dx'\right)\left(\int xe^{-ipx/\hbar}\Psi(x)dx\right)dp \tag{6}$$

$$=\frac{1}{2\pi\hbar}\int\left(\int\int e^{ip(x'-x)/\hbar}\Psi^{*}(x')x\Psi(x)dx'dx\right)dp \tag{7}$$

We can now do the integral over p first, and use the dodgy formula we obtained earlier, which expressed a delta function as an integral over a complex exponential:

$$\frac{1}{2\pi\hbar}\int e^{ip(x'-x)/\hbar}dp = \delta(x'-x) \tag{8}$$

Thus:

$$\int \Phi^* \left(-\frac{\hbar}{i}\frac{\partial}{\partial p}\right)\Phi dp = \int\int \delta(x'-x)\Psi^*(x')x\Psi(x)dx'dx \tag{9}$$

$$= \int \Psi^*(x)x\Psi(x)dx \tag{10}$$

$$= \langle x\rangle \tag{11}$$

QED.