COMMUTATORS: A FEW THEOREMS

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 3.13.

The commutator of two operators is defined as

$$[A,B] \equiv AB - BA \tag{1}$$

In general, a commutator is non-zero, since the order in which we apply operators can make a difference. In practice, to work out a commutator we need to apply it to a test function f, so that we really need to work out $[A,B]f$ and then remove the test function to see the result. This is because many operators, such as the momentum, involve taking the derivative.

We'll now have a look at a few theorems involving commutators.

Theorem 1:

$$[AB,C] = A[B,C] + [A,C]B \tag{2}$$

Proof: The LHS is:

$$[AB,C] = ABC - CAB \tag{3}$$

The RHS is:

$$\begin{aligned} A[B,C] + [A,C]B &= ABC - ACB + ACB - CAB & (4)\\ &= ABC - CAB & (5)\\ &= [AB,C] & (6) \end{aligned}$$

QED.

Theorem 2:

$$[x^n, p] = i\hbar n x^{n-1} \tag{7}$$

where p is the momentum operator.

Proof: Using $p = \frac{\hbar}{i}\partial/\partial x$ and letting the commutator operate on some arbitrary function g:

$$\begin{aligned}
[x^n,p]\,g &= x^n\frac{\hbar}{i}\frac{\partial g}{\partial x}-\frac{\hbar}{i}\frac{\partial}{\partial x}(x^n g) && (8)\\
&= x^n\frac{\hbar}{i}\frac{\partial g}{\partial x}-\frac{\hbar}{i}nx^{n-1}g-x^n\frac{\hbar}{i}\frac{\partial g}{\partial x} && (9)\\
&= i\hbar nx^{n-1}g && (10)
\end{aligned}$$

Removing the function g gives the result $[x^n,p]=i\hbar nx^{n-1}$. QED.

Theorem 3:

$$[f(x),p]=i\hbar\frac{df}{dx} \tag{11}$$

Again, letting the commutator operate on a function g:

$$\begin{aligned}
[f(x),p] &= f\frac{\hbar}{i}\frac{\partial g}{\partial x}-\frac{\hbar}{i}\frac{\partial}{\partial x}(fg) && (12)\\
&= f\frac{\hbar}{i}\frac{\partial g}{\partial x}-\frac{\hbar}{i}\frac{\partial f}{\partial x}g-f\frac{\hbar}{i}\frac{\partial g}{\partial x} && (13)\\
&= ih\frac{\partial f}{\partial x}g && (14)
\end{aligned}$$

Removing g gives the result $[f(x),p]=ih\partial f/\partial x$. QED.

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