HERMITIAN OPERATORS: COMMON EIGENFUNCTIONS IMPLIES THEY COMMUTE

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 3.15.

The eigenfunctions of a hermitian operator form a complete set, in the sense that any function can be expressed as a linear combination of these eigenfunctions. A consequence of this is the theorem that, if two hermitian operators have the *same* set of eigenfunctions (though possibly with different eigenvalues), then these two operators must commute.

If the two operators $\hat{P}$ and $\hat{Q}$ have the same complete set of common eigenfunctions, then a function f in Hilbert space can be written as a series in terms of these eigenfunctions

$$f = \sum c_n f_n \tag{0.1}$$

where f_n are the eigenfunctions, and c_n are the coefficients.

Now for operator $\hat{P}$ we have $\hat{P} f_n = p_n f_n$ where p_n is the eigenvalue when $\hat{P}$ operates on f_n. Similarly for $\hat{Q}$ we have $\hat{Q} f_n = q_n f_n$. Since the two operators share the same set of eigenfunctions, applying the operators in either order to f will result in each term in the series being multiplied by the product of the two eigenvalues for each of the operators:

$$\hat{P}\hat{Q} f = \hat{Q}\hat{P} f = \sum c_n p_n q_n f_n \tag{0.2}$$

Thus the commutator will be zero if two operators share the same set of eigenfunctions. QED.