

UNCERTAINTY PRINCIPLE: CONDITION FOR MINIMUM UNCERTAINTY

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Section 3.5.2; Problem 3.16.

The generalized uncertainty principle relating two operators A and B is

$$(0.1) \quad \sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [A, B] \rangle \right)^2$$

where $[A, B] = AB - BA$ is the commutator of the operators.

This relation was derived using two inequalities. The first is the Schwarz inequality which, in bra-ket notation is

$$(0.2) \quad \langle f|f \rangle \langle g|g \rangle \geq |\langle f|g \rangle|^2$$

The second inequality is condition on a complex number $z = x + iy$:

$$(0.3) \quad |z|^2 \geq y^2$$

It's interesting to see what happens if we require these two relations to be equalities rather than inequalities. This will give us a condition on the functions f and g that gives the minimum uncertainty between them.

In the case of the Schwarz inequality, we want

$$(0.4) \quad \langle f|f \rangle \langle g|g \rangle = |\langle f|g \rangle|^2$$

To see what this implies, we can examine the proof of the Schwarz inequality. To this end, we'll introduce the function

$$(0.5) \quad |h\rangle \equiv |g\rangle - \frac{\langle f|g \rangle}{\langle f|f \rangle} |f\rangle$$

Since $\langle h|h \rangle \geq 0$

$$(0.6) \quad \langle h|h \rangle = \langle g|g \rangle - \frac{\langle f|g \rangle}{\langle f|f \rangle} \langle g|f \rangle - \frac{\langle g|f \rangle}{\langle f|f \rangle} \langle f|g \rangle + \left(\frac{|\langle f|g \rangle|}{\langle f|f \rangle} \right)^2 \langle f|f \rangle$$

$$(0.7) \quad = \langle g|g \rangle - \frac{|\langle f|g \rangle|^2}{\langle f|f \rangle}$$

$$(0.8) \quad \geq 0$$

$$(0.9) \quad \langle f|f \rangle \langle g|g \rangle \geq |\langle f|g \rangle|^2$$

In order for this inequality to be replaced with an equality, we would need

$$(0.10) \quad |h \rangle = 0$$

$$(0.11) \quad |g \rangle = \frac{\langle f|g \rangle}{\langle f|f \rangle} |f \rangle$$

That is, the Schwarz inequality becomes an equality if one function is a scalar multiple of the other:

$$(0.12) \quad |g \rangle = c |f \rangle$$

where c is, in general, a complex scalar.

The second inequality (0.3 above) is an equality if $x = 0$ so that z is purely imaginary. In our original derivation of the uncertainty principle, we used $z = \langle f|g \rangle$, so if we're requiring equality, we get

$$(0.13) \quad z = \langle f|g \rangle$$

$$(0.14) \quad = c \langle f|f \rangle$$

and we require this to be purely imaginary. Since $\langle f|f \rangle$ is always real, this means that c must be imaginary, so we can write that the condition for equality is

$$(0.15) \quad |g \rangle = ia |f \rangle$$

with a a real scalar.

In terms of the operators A and B , referring back to our original derivation, this means that in order to get the minimum uncertainty, the wave function Ψ must satisfy

$$(0.16) \quad |(A - \langle A \rangle)\Psi\rangle = ia|(B - \langle B \rangle)\Psi\rangle$$

For example, if we consider the position and momentum operators, we get

$$(0.17) \quad \left(\frac{\hbar}{i} \frac{d}{dx} - \langle p \rangle \right) \Psi = ia(x - \langle x \rangle) \Psi$$

We can put the derivative on the LHS and everything else on the RHS (note that $\langle x \rangle$ and $\langle p \rangle$ are constants):

$$(0.18) \quad \frac{d\Psi}{dx} = \frac{i}{\hbar} (ia(x - \langle x \rangle) + \langle p \rangle) \Psi$$

$$(0.19) \quad \frac{d\Psi}{\Psi} = \left(-\frac{a}{\hbar} (x - \langle x \rangle) + \frac{i\langle p \rangle}{\hbar} \right) dx$$

$$(0.20) \quad \ln \Psi = -\frac{a}{2\hbar} (x - \langle x \rangle)^2 + \frac{i\langle p \rangle x}{\hbar} + \ln A$$

$$(0.21) \quad \Psi = A e^{-a(x - \langle x \rangle)^2 / 2\hbar} e^{i\langle p \rangle x / \hbar}$$

In any stationary state $\langle p \rangle = 0$, so any system in which there is a stationary state that has a gaussian wave function will have minimum position-momentum uncertainty. One case where this occurs is the ground state of the harmonic oscillator. (All higher states of the harmonic oscillator are gaussians multiplied by a higher-degree Hermite polynomial, so they aren't pure gaussians and will therefore have a higher uncertainty.)

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