

## UNCERTAINTY PRINCIPLE: CONDITION FOR MINIMUM UNCERTAINTY

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Post date: 21 Sep 2012.

References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Section 3.5.2; Problem 3.16.

The generalized uncertainty principle relating two operators  $A$  and  $B$  is

$$\sigma_A^2 \sigma_B^2 \geq \left( \frac{1}{2i} \langle [A, B] \rangle \right)^2 \quad (1)$$

where  $[A, B] = AB - BA$  is the commutator of the operators.

This relation was derived using two inequalities. The first is the Schwarz inequality which, in bra-ket notation is

$$\langle f|f \rangle \langle g|g \rangle \geq |\langle f|g \rangle|^2 \quad (2)$$

The second inequality is condition on a complex number  $z = x + iy$ :

$$|z|^2 \geq y^2 \quad (3)$$

It's interesting to see what happens if we require these two relations to be equalities rather than inequalities. This will give us a condition on the functions  $f$  and  $g$  that gives the minimum uncertainty between them.

In the case of the Schwarz inequality, we want

$$\langle f|f \rangle \langle g|g \rangle = |\langle f|g \rangle|^2 \quad (4)$$

To see what this implies, we can examine the proof of the Schwarz inequality. To this end, we'll introduce the function

$$|h\rangle \equiv |g\rangle - \frac{\langle f|g \rangle}{\langle f|f \rangle} |f\rangle \quad (5)$$

Since  $\langle h|h \rangle \geq 0$

$$\langle h|h \rangle = \langle g|g \rangle - \frac{\langle f|g \rangle}{\langle f|f \rangle} \langle g|f \rangle - \frac{\langle g|f \rangle}{\langle f|f \rangle} \langle f|g \rangle + \left( \frac{|\langle f|g \rangle|}{\langle f|f \rangle} \right)^2 \langle f|f \rangle \quad (6)$$

$$= \langle g|g \rangle - \frac{|\langle f|g \rangle|^2}{\langle f|f \rangle} \quad (7)$$

$$\geq 0 \quad (8)$$

$$\langle f|f \rangle \langle g|g \rangle \geq |\langle f|g \rangle|^2 \quad (9)$$

In order for this inequality to be replaced with an equality, we would need

$$|h \rangle = 0 \quad (10)$$

$$|g \rangle = \frac{\langle f|g \rangle}{\langle f|f \rangle} |f \rangle \quad (11)$$

That is, the Schwarz inequality becomes an equality if one function is a scalar multiple of the other:

$$|g \rangle = c |f \rangle \quad (12)$$

where  $c$  is, in general, a complex scalar.

The second inequality (3 above) is an equality if  $x = 0$  so that  $z$  is purely imaginary. In our original derivation of the uncertainty principle, we used  $z = \langle f|g \rangle$ , so if we're requiring equality, we get

$$z = \langle f|g \rangle \quad (13)$$

$$= c \langle f|f \rangle \quad (14)$$

and we require this to be purely imaginary. Since  $\langle f|f \rangle$  is always real, this means that  $c$  must be imaginary, so we can write that the condition for equality is

$$|g \rangle = ia |f \rangle \quad (15)$$

with  $a$  a real scalar.

In terms of the operators  $A$  and  $B$ , referring back to our original derivation, this means that in order to get the minimum uncertainty, the wave function  $\Psi$  must satisfy

$$|(A - \langle A \rangle)\Psi \rangle = ia |(B - \langle B \rangle)\Psi \rangle \quad (16)$$

For example, if we consider the position and momentum operators, we get

$$\left(\frac{\hbar}{i} \frac{d}{dx} - \langle p \rangle\right) \Psi = ia(x - \langle x \rangle) \Psi \quad (17)$$

We can put the derivative on the LHS and everything else on the RHS (note that  $\langle x \rangle$  and  $\langle p \rangle$  are constants):

$$\frac{d\Psi}{dx} = \frac{i}{\hbar}(ia(x - \langle x \rangle) + \langle p \rangle) \Psi \quad (18)$$

$$\frac{d\Psi}{\Psi} = \left(-\frac{a}{\hbar}(x - \langle x \rangle) + \frac{i\langle p \rangle}{\hbar}\right) dx \quad (19)$$

$$\ln \Psi = -\frac{a}{2\hbar}(x - \langle x \rangle)^2 + \frac{i\langle p \rangle x}{\hbar} + \ln A \quad (20)$$

$$\Psi = Ae^{-a(x - \langle x \rangle)^2/2\hbar} e^{i\langle p \rangle x/\hbar} \quad (21)$$

In any stationary state  $\langle p \rangle = 0$ , so any system in which there is a stationary state that has a gaussian wave function will have minimum position-momentum uncertainty. One case where this occurs is the ground state of the harmonic oscillator. (All higher states of the harmonic oscillator are gaussians multiplied by a higher-degree Hermite polynomial, so they aren't pure gaussians and will therefore have a higher uncertainty.)

#### PINGBACKS

Pingback: Harmonic oscillator - zero-point energy from uncertainty principle