

UNCERTAINTY PRINCIPLE: RATES OF CHANGE OF OPERATORS

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 3.17.

The rate of change of the expectation value of an operator Q is

$$(1) \quad \frac{d}{dt} \langle Q \rangle = \frac{i}{\hbar} \langle [H, Q] \rangle + \left\langle \frac{\partial Q}{\partial t} \right\rangle$$

where H is the hamiltonian of the system (assumed time-independent).

We'll look at a few applications of this formula.

(a) With $Q = 1$, since any operator commutes with a constant, and Q has no explicit time dependence,

$$(2) \quad \frac{d\langle Q \rangle}{dt} = 0$$

This is another way of stating the fact that the integral of the square modulus of the wave function remains 1 for all time (i.e. the square modulus of the wave function remains as a probability density), since the integral of the square modulus is essentially the "expectation value of the unit operator".

(b) With $Q = H$, since any operator commutes with itself, we have

$$(3) \quad \frac{d\langle H \rangle}{dt} = \left\langle \frac{\partial H}{\partial t} \right\rangle$$

If the energy has no explicit time dependence (that is, the potential function does not depend on time), then $\partial H / \partial t = 0$ and this result expresses the conservation of energy.

(c) With $Q = x$, we can use the commutator worked out earlier:

$$(4) \quad [\hat{H}, \hat{x}] = -\frac{i\hbar}{m} p = -\frac{i\hbar}{m} \frac{\partial}{\partial x}$$

Since x and t are independent variables, $\partial x / \partial t = 0$, so we get:

$$(5) \quad \frac{d\langle x \rangle}{dt} = \frac{i}{\hbar} \langle [\hat{H}, \hat{x}] \rangle = \frac{\langle p \rangle}{m}$$

This is the quantum equivalent of the classical equation $p = mv$.

(d) With $Q = p$, we need to work out the commutator $[\hat{H}, \hat{p}]$. Using an auxiliary function g on which this commutator can operate, we get:

$$(6) \quad [\hat{H}, \hat{p}]g = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V, \frac{\hbar}{i} \frac{\partial}{\partial x} \right] g$$

$$(7) \quad = \frac{\hbar}{i} \left(V \frac{\partial g}{\partial x} - \frac{\partial}{\partial x} (Vg) \right)$$

$$(8) \quad = -\frac{\hbar}{i} \frac{\partial V}{\partial x} g$$

Therefore, the commutator is $[\hat{H}, \hat{p}] = -\frac{\hbar}{i} \frac{\partial V}{\partial x}$, and we get

$$(9) \quad \frac{d\langle p \rangle}{dt} = - \left\langle \frac{\partial V}{\partial x} \right\rangle$$

This is Ehrenfest's theorem.

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