UNCERTAINTY PRINCIPLE: RATES OF CHANGE OF OPERATORS

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 3.17.

The rate of change of the expectation value of an operator Q is

$$\frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\langle[H,Q]\rangle + \left\langle\frac{\partial Q}{\partial t}\right\rangle \tag{1}$$

where H is the hamiltonian of the system (assumed time-independent).

We'll look at a few applications of this formula.

(a) With $Q=1$, since any operator commutes with a constant, and Q has no explicit time dependence,

$$\frac{d\langle Q\rangle}{dt} = 0 \tag{2}$$

This is another way of stating the fact that the integral of the square modulus of the wave function remains 1 for all time (i.e. the square modulus of the wave function remains as a probability density), since the integral of the square modulus is essentially the "expectation value of the unit operator".

(b) With $Q=H$, since any operator commutes with itself, we have

$$\frac{d\langle H\rangle}{dt} = \left\langle\frac{\partial H}{\partial t}\right\rangle \tag{3}$$

If the energy has no explicit time dependence (that is, the potential function does not depend on time), then $\partial H/\partial t = 0$ and this result expresses the conservation of energy.

(c) With $Q=x$, we can use the commutator worked out earlier:

$$[\hat{H},\hat{x}] = -\frac{i\hbar}{m}p = -\frac{i\hbar}{m}\frac{\partial}{\partial x} \tag{4}$$

Since x and t are independent variables, $\partial x/\partial t = 0$, so we get:

$$\frac{d\langle x\rangle}{dt} = \frac{i}{\hbar}\langle[\hat{H},\hat{x}]\rangle = \frac{\langle p\rangle}{m} \tag{5}$$

This is the quantum equivalent of the classical equation $p=mv$.

(d) With $Q=p$, we need to work out the commutator $[\hat{H},\hat{p}]$. Using an auxiliary function g on which this commutator can operate, we get:

$$[\hat{H},\hat{p}]g = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V,\frac{\hbar}{i}\frac{\partial}{\partial x}\right]g \tag{6}$$

$$= \frac{\hbar}{i}\left(V\frac{\partial g}{\partial x}-\frac{\partial}{\partial x}(Vg)\right) \tag{7}$$

$$= -\frac{\hbar}{i}\frac{\partial V}{\partial x}g \tag{8}$$

Therefore, the commutator is $[\hat{H},\hat{p}]=-\frac{\hbar}{i}\frac{\partial V}{\partial x}$, and we get

$$\frac{d\langle p\rangle}{dt}=-\left\langle\frac{\partial V}{\partial x}\right\rangle \tag{9}$$

This is Ehrenfest's theorem.

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