UNCERTAINTY PRINCIPLE: RATES OF CHANGE OF OPERATORS

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The rate of change of the expectation value of an operator $Q$ is

$$\frac{d}{dt} \langle Q \rangle = \frac{i}{\hbar} \langle [H, Q] \rangle + \langle \frac{\partial Q}{\partial t} \rangle$$

(1)

where $H$ is the hamiltonian of the system (assumed time-independent).

We’ll look at a few applications of this formula.

(a) With $Q = 1$, since any operator commutes with a constant, and $Q$ has no explicit time dependence,

$$\frac{d\langle Q \rangle}{dt} = 0$$

(2)

This is another way of stating the fact that the integral of the square modulus of the wave function remains 1 for all time (i.e. the square modulus of the wave function remains as a probability density), since the integral of the square modulus is essentially the “expectation value of the unit operator”.

(b) With $Q = H$, since any operator commutes with itself, we have

$$\frac{d\langle H \rangle}{dt} = \langle \frac{\partial H}{\partial t} \rangle$$

(3)

If the energy has no explicit time dependence (that is, the potential function does not depend on time), then $\partial H/\partial t = 0$ and this result expresses the conservation of energy.

(c) With $Q = x$, we can use the commutator worked out earlier

$$[\hat{H}, \hat{x}] = \frac{ih}{m} \hat{p} = -\frac{ih}{m} \frac{\partial}{\partial x}$$

(4)

Since $x$ and $t$ are independent variables, $\partial x/\partial t = 0$, so we get:

$$\frac{d\langle x \rangle}{dt} = \frac{i}{\hbar} \langle [\hat{H}, \hat{x}] \rangle = \frac{\langle p \rangle}{m}$$

(5)
This is the quantum equivalent of the classical equation \( p = mv \).

(d) With \( Q = p \), we need to work out the commutator \([\hat{H}, \hat{p}]\). Using an auxiliary function \( g \) on which this commutator can operate, we get:

\[
[\hat{H}, \hat{p}] g = \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V, \frac{\hbar}{i} \frac{\partial}{\partial x} \right] g = \frac{\hbar}{i} \left( V \frac{\partial g}{\partial x} - \frac{\partial}{\partial x} (V g) \right) = -\frac{\hbar}{i} \frac{\partial V}{\partial x} g
\]

Therefore, the commutator is \([\hat{H}, \hat{p}] = -\frac{\hbar}{i} \frac{\partial V}{\partial x} \), and we get

\[
\frac{d\langle p \rangle}{dt} = -\langle \frac{\partial V}{\partial x} \rangle
\]

This is **Ehrenfest’s theorem**.