

## ENERGY-TIME UNCERTAINTY PRINCIPLE: INFINITE SQUARE WELL

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 3.18.

As an example of the energy-time uncertainty relation, we can look again at the example of a particle in the infinite square well, which starts off in a combination of the two lowest states:

$$(1) \quad \Psi(x, 0) = A [\psi_1(x) + \psi_2(x)]$$

We would like to verify by explicit calculation that

$$(2) \quad \sigma_H \sigma_x \geq \frac{\hbar}{2} \frac{d\langle x \rangle}{dt}$$

Referring back to our calculations on that example, we can get some of the needed quantities immediately:

$$(3) \quad \frac{d\langle x \rangle}{dt} = \frac{8\hbar}{3ma} \sin(3\omega t)$$

$$(4) \quad \langle H \rangle = \frac{5\pi^2 \hbar^2}{4ma^2}$$

$$(5) \quad \langle x \rangle = \frac{a}{2} - \frac{16a}{9\pi^2} \cos(3\omega t)$$

$$(6) \quad \Psi(x, t) = \frac{\sqrt{2}}{2} \psi_1(x) e^{-i\omega t} + \frac{\sqrt{2}}{2} \psi_2(x) e^{-4i\omega t}$$

$$(7) \quad E_i = \frac{n^2 \pi^2 \hbar^2}{2ma^2}$$

$$(8) \quad \psi_n(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}$$

We can calculate  $\sigma_H^2$  using these equations:

$$(9) \quad (\hat{H} - \langle H \rangle)\Psi = \frac{\sqrt{2}}{2} [(E_1 - \langle H \rangle)\psi_1 e^{-i\omega t} + (E_2 - \langle H \rangle)\psi_2 e^{-4i\omega t}]$$

$$(10) \quad \sigma_H^2 = \langle (\hat{H} - \langle H \rangle)\Psi | (\hat{H} - \langle H \rangle)\Psi \rangle$$

$$(11) \quad = \frac{1}{2} [(E_1 - \langle H \rangle)^2 + (E_2 - \langle H \rangle)^2]$$

$$(12) \quad = \frac{9}{16} \left( \frac{\pi^2 \hbar^2}{ma^2} \right)^2$$

Thus

$$(13) \quad \sigma_H = \frac{3}{4} \frac{\pi^2 \hbar^2}{ma^2}$$

To get  $\sigma_x$ , we can use the fact that, for any quantity  $A$ ,  $\sigma_A^2 = \langle A^2 \rangle - \langle A \rangle^2$ . We already know  $\langle x \rangle$ , so we can calculate  $\langle x^2 \rangle$  using integration. We get:

$$(14) \quad \langle x^2 \rangle = \frac{1}{2} \int_0^a (x^2 \psi_1^2(x) + x^2 \psi_2^2(x) + 2x^2 \psi_1 \psi_2 \cos(3\omega t)) dx$$

$$(15) \quad = \frac{a^2}{144\pi^2} (48\pi^2 - 45 - 256 \cos(3\omega t))$$

where we did the integral using Maple. We now get  $\sigma_x$  (after simplifying):

$$(16) \quad \sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2$$

$$(17) \quad = \frac{a^2}{1296\pi^4} (108\pi^4 - 405\pi^2 - 4096 \cos^2(3\omega t))$$

$$(18) \quad = a^2 \left[ \frac{1}{12} - \frac{5}{16\pi^2} - \left( \frac{16}{9\pi^2} \cos 3\omega t \right)^2 \right]$$

If we define

$$(19) \quad \beta^2 \equiv 108\pi^4 - 405\pi^2 - 4096 \cos^2(3\omega t)$$

then

$$(20) \quad \sigma_x = \frac{a}{36\pi^2} \beta$$

and

$$(21) \quad \sigma_H \sigma_x = \frac{1}{48} \frac{\hbar^2}{ma} \beta$$

We would like to show that this satisfies the inequality

$$(22) \quad \frac{1}{48} \frac{\hbar^2}{ma} \beta \geq \frac{\hbar}{2} \frac{d\langle x \rangle}{dt} = \frac{4\hbar^2}{3ma} \sin(3\omega t)$$

which will be true if  $\beta \geq 64 \sin(3\omega t)$ , or  $\beta^2 \geq 4096 \sin^2(3\omega t)$ . Substituting for  $\beta$  from 19 we get

$$(23) \quad 108\pi^4 - 405\pi^2 - 4096 \cos^2(3\omega t) \stackrel{?}{\geq} 4096 \sin^2(3\omega t)$$

$$(24) \quad 108\pi^4 - 405\pi^2 \stackrel{?}{\geq} 4096 \cos^2(3\omega t) + 4096 \sin^2(3\omega t)$$

$$(25) \quad 108\pi^4 - 405\pi^2 \stackrel{?}{\geq} 4096$$

Putting in the numbers on the LHS leads to  $6522.992 > 4096$ , which is true so the uncertainty principle is satisfied for this case.