ENERGY-TIME UNCERTAINTY PRINCIPLE: INFINITE SQUARE WELL

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 3.18.

As an example of the energy-time uncertainty relation, we can look again at the example of a particle in the infinite square well, which starts off in a combination of the two lowest states:

$$\Psi(x,0)=A\left[\psi_1(x)+\psi_2(x)\right] \tag{0.1}$$

We would like to verify by explicit calculation that

$$\sigma_H\sigma_x\ge\frac{\hbar}{2}\frac{d\langle x\rangle}{dt} \tag{0.2}$$

Referring back to our calculations on that example, we can get some of the needed quantities immediately:

$$\begin{aligned}
\frac{d\langle x\rangle}{dt} &= \frac{8\hbar}{3ma}\sin(3\omega t) && (0.3)\\
\langle H\rangle &= \frac{5\pi^2\hbar^2}{4ma^2} && (0.4)\\
\langle x\rangle &= \frac{a}{2}-\frac{16a}{9\pi^2}\cos(3\omega t) && (0.5)\\
\Psi(x,t) &= \frac{\sqrt{2}}{2}\psi_1(x)e^{-i\omega t}+\frac{\sqrt{2}}{2}\psi_2(x)e^{-4i\omega t} && (0.6)\\
E_i &= \frac{n^2\pi^2\hbar^2}{2ma^2} && (0.7)\\
\psi_n(x) &= \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a} && (0.8)
\end{aligned}$$

We can calculate σ_H^2 using these equations:

$$(\hat{H}-\langle H\rangle)\Psi = \frac{\sqrt{2}}{2}[(E_1-\langle H\rangle])\psi_1 e^{-i\omega t}+(E_2-\langle H\rangle)\psi_2 e^{-4i\omega t}] \tag{0.9}$$

$$\sigma_H^2 = \langle(\hat{H}-\langle H\rangle)\Psi|(\hat{H}-\langle H\rangle)\Psi\rangle \tag{0.10}$$

$$= \frac{1}{2}[(E_1-\langle H\rangle])^2+(E_2-\langle H\rangle)^2] \tag{0.11}$$

$$= \frac{9}{16}\left(\frac{\pi^2\hbar^2}{ma^2}\right)^2 \tag{0.12}$$

Thus

$$\sigma_H = \frac{3}{4}\frac{\pi^2\hbar^2}{ma^2} \tag{0.13}$$

To get σ_x, we can use the fact that, for any quantity A, $\sigma_A^2 = \langle A^2\rangle - \langle A\rangle^2$. We already know $\langle x\rangle$, so we can calculate $\langle x^2\rangle$ using integration. We get:

$$\langle x^2\rangle = \frac{1}{2}\int_0^a (x^2\psi_1^2(x)+x^2\psi_2^2(x)+2x^2\psi_1\psi_2\cos(3\omega t))dx \tag{0.14}$$

$$= \frac{a^2}{144\pi^2}(48\pi^2-45-256\cos(3\omega t)) \tag{0.15}$$

where we did the integral using Maple. We now get σ_x (after simplifying):

$$\sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2 \tag{0.16}$$

$$= \frac{a^2}{1296\pi^4}(108\pi^4-405\pi^2-4096\cos^2(3\omega t)) \tag{0.17}$$

$$= a^2\left[\frac{1}{12}-\frac{5}{16\pi^2}-\left(\frac{16}{9\pi^2}\cos 3\omega t\right)^2\right] \tag{0.18}$$

If we define

$$\beta^2 \equiv 108\pi^4-405\pi^2-4096\cos^2(3\omega t) \tag{0.19}$$

then

$$\sigma_x = \frac{a}{36\pi^2}\beta \tag{0.20}$$

and

$$\sigma_H \sigma_x = \frac{1}{48}\frac{\hbar^2}{ma}\beta \tag{0.21}$$

We would like to show that this satisfies the inequality

$$\frac{1}{48}\frac{\hbar^2}{ma}\beta \geq \frac{\hbar}{2}\frac{d\langle x\rangle}{dt} = \frac{4\hbar^2}{3ma}\sin(3\omega t) \tag{0.22}$$

which will be true if $\beta \geq 64\sin(3\omega t)$, or $\beta^2 \geq 4096\sin^2(3\omega t)$. Substituting for β from 0.19 we get

$$108\pi^4 - 405\pi^2 - 4096\cos^2(3\omega t) \stackrel{?}{\geq} 4096\sin^2(3\omega t) \tag{0.23}$$

$$108\pi^4 - 405\pi^2 \stackrel{?}{\geq} 4096\cos^2(3\omega t) + 4096\sin^2(3\omega t) \tag{0.24}$$

$$108\pi^4 - 405\pi^2 \stackrel{?}{\geq} 4096 \tag{0.25}$$

Putting in the numbers on the LHS leads to $6522.992 > 4096$, which is true so the uncertainty principle is satisfied for this case.