

ENERGY-TIME UNCERTAINTY PRINCIPLE: INFINITE SQUARE WELL

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 3.18.

As an example of the energy-time uncertainty relation, we can look again at the example of a particle in the infinite square well, which starts off in a combination of the two lowest states:

$$\Psi(x, 0) = A [\psi_1(x) + \psi_2(x)] \quad (1)$$

We would like to verify by explicit calculation that

$$\sigma_H \sigma_x \geq \frac{\hbar}{2} \frac{d\langle x \rangle}{dt} \quad (2)$$

Referring back to our calculations on that example, we can get some of the needed quantities immediately:

$$\frac{d\langle x \rangle}{dt} = \frac{8\hbar}{3ma} \sin(3\omega t) \quad (3)$$

$$\langle H \rangle = \frac{5\pi^2 \hbar^2}{4ma^2} \quad (4)$$

$$\langle x \rangle = \frac{a}{2} - \frac{16a}{9\pi^2} \cos(3\omega t) \quad (5)$$

$$\Psi(x, t) = \frac{\sqrt{2}}{2} \psi_1(x) e^{-i\omega t} + \frac{\sqrt{2}}{2} \psi_2(x) e^{-4i\omega t} \quad (6)$$

$$E_i = \frac{n^2 \pi^2 \hbar^2}{2ma^2} \quad (7)$$

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} \quad (8)$$

We can calculate σ_H^2 using these equations:

$$(\hat{H} - \langle H \rangle)\Psi = \frac{\sqrt{2}}{2}[(E_1 - \langle H \rangle)]\psi_1 e^{-i\omega t} + (E_2 - \langle H \rangle)\psi_2 e^{-4i\omega t} \quad (9)$$

$$\sigma_H^2 = \langle (\hat{H} - \langle H \rangle)\Psi | (\hat{H} - \langle H \rangle)\Psi \rangle \quad (10)$$

$$= \frac{1}{2}[(E_1 - \langle H \rangle)]^2 + (E_2 - \langle H \rangle)^2 \quad (11)$$

$$= \frac{9}{16} \left(\frac{\pi^2 \hbar^2}{ma^2} \right)^2 \quad (12)$$

Thus

$$\sigma_H = \frac{3}{4} \frac{\pi^2 \hbar^2}{ma^2} \quad (13)$$

To get σ_x , we can use the fact that, for any quantity A , $\sigma_A^2 = \langle A^2 \rangle - \langle A \rangle^2$. We already know $\langle x \rangle$, so we can calculate $\langle x^2 \rangle$ using integration. We get:

$$\langle x^2 \rangle = \frac{1}{2} \int_0^a (x^2 \psi_1^2(x) + x^2 \psi_2^2(x) + 2x^2 \psi_1 \psi_2 \cos(3\omega t)) dx \quad (14)$$

$$= \frac{a^2}{144\pi^2} (48\pi^2 - 45 - 256 \cos(3\omega t)) \quad (15)$$

where we did the integral using Maple. We now get σ_x (after simplifying):

$$\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2 \quad (16)$$

$$= \frac{a^2}{1296\pi^4} (108\pi^4 - 405\pi^2 - 4096 \cos^2(3\omega t)) \quad (17)$$

$$= a^2 \left[\frac{1}{12} - \frac{5}{16\pi^2} - \left(\frac{16}{9\pi^2} \cos 3\omega t \right)^2 \right] \quad (18)$$

If we define

$$\beta^2 \equiv 108\pi^4 - 405\pi^2 - 4096 \cos^2(3\omega t) \quad (19)$$

then

$$\sigma_x = \frac{a}{36\pi^2} \beta \quad (20)$$

and

$$\sigma_H \sigma_x = \frac{1}{48} \frac{\hbar^2}{ma} \beta \quad (21)$$

We would like to show that this satisfies the inequality

$$\frac{1}{48} \frac{\hbar^2}{ma} \beta \geq \frac{\hbar}{2} \frac{d\langle x \rangle}{dt} = \frac{4\hbar^2}{3ma} \sin(3\omega t) \quad (22)$$

which will be true if $\beta \geq 64 \sin(3\omega t)$, or $\beta^2 \geq 4096 \sin^2(3\omega t)$. Substituting for β from 19 we get

$$108\pi^4 - 405\pi^2 - 4096 \cos^2(3\omega t) \stackrel{?}{\geq} 4096 \sin^2(3\omega t) \quad (23)$$

$$108\pi^4 - 405\pi^2 \stackrel{?}{\geq} 4096 \cos^2(3\omega t) + 4096 \sin^2(3\omega t) \quad (24)$$

$$108\pi^4 - 405\pi^2 \stackrel{?}{\geq} 4096 \quad (25)$$

Putting in the numbers on the LHS leads to $6522.992 > 4096$, which is true so the uncertainty principle is satisfied for this case.