# ENERGY-TIME UNCERTAINTY PRINCIPLE: GAUSSIAN FREE PARTICLE

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 3.19.

As another example of the energy-time uncertainty relation, we can look again at the example of a travelling free particle with a Gaussian wave packet. We have already worked out most of what we need to test the uncertainty relation:

$$\begin{aligned}\langle x\rangle &= \frac{l\hbar t}{m} &(1)\\ \langle x^2\rangle &= \frac{1+(2a\hbar t/m)^2+a(2\hbar lt/m)^2}{4a} &(2)\\ \langle p^2\rangle &= \hbar^2(a+l^2) &(3)\end{aligned}$$

From this we get

$$\begin{aligned}\langle H\rangle &= \frac{\langle p^2\rangle}{2m} &(4)\\ &= \frac{\hbar^2(a+l^2)}{2m} &(5)\end{aligned}$$

We still need $\langle H^2\rangle$. From our previous calculations, we have the wave function:

$$\Psi(x,t)=\left(\frac{2a}{\pi}\right)^{1/4}\frac{e^{\left(-ax^2+i(lx-\hbar l^2t/2m)\right)/(1+2i\hbar at/m)}}{\sqrt{1+2i\hbar at/m}} \qquad (6)$$

We can calculate $\langle H^2\rangle$ by direct integration, using Maple:

$$\begin{aligned}\langle H^2\rangle &= \frac{\hbar^4}{4m^2}\int_{-\infty}^{\infty}\left|\frac{d^2\Psi(x,t)}{dx^2}\right|^2 dx &(7)\\ &= \frac{\hbar^4}{4m^2}\left(3a^2+6al^2+l^4\right) &(8)\end{aligned}$$

As a check on this result, we can work out the units (always a good test to make sure you haven't dropped a factor somewhere). From the original wave function, since exponents must be dimensionless, we know that $a$ has dimensions $distance^{-2}$ and $l$ has $distance^{-1}$. Planck's constant has dimensions of $energy \times time$, so the expression above has overall units of $energy^4 \times time^4 \times mass^{-2} \times distance^{-4} = energy^2$. (Recall kinetic energy is $mv^2/2$.) It's also worth noting that $\left\langle H^2\right\rangle$ is independent of time.

From here, we can get $\sigma_H^2$:

$$\begin{aligned}
\sigma_H^2 &= \left\langle H^2\right\rangle - \langle H\rangle^2 && (9)\\
&= \frac{a\hbar^4}{2m^2}\left(a+2l^2\right) && (10)
\end{aligned}$$

We also have, from above

$$\begin{aligned}
\sigma_x^2 &= \left\langle x^2\right\rangle - \langle x\rangle^2 && (11)\\
&= \frac{m^2+4\left(at\hbar\right)^2}{4am^2} && (12)
\end{aligned}$$

We're trying to show that $\sigma_H\sigma_x \geq \frac{\hbar}{2}\frac{d\langle x\rangle}{dt} = l\hbar^2/2m$ from above. So we want

$$\begin{aligned}
\sigma_H^2\sigma_x^2 &\geq \frac{l^2\hbar^4}{4m^2} && (13)\\
\frac{a\hbar^4}{2m^2}\left(a+2l^2\right)\frac{m^2+4\left(at\hbar\right)^2}{4am^2} &\geq \frac{l^2\hbar^4}{4m^2} && (14)
\end{aligned}$$

The minimum of the LHS occurs at $t=0$ so if the inequality is true there, it is true always. In this case, it reduces to

$$\begin{aligned}
a+2l^2 &\geq 2l^2 && (15)\\
a &\geq 0 && (16)
\end{aligned}$$

This final condition is certainly true (it is required for the Gaussian wave form to converge at large $x$), so the uncertainty condition is verified.