

ENERGY-TIME UNCERTAINTY PRINCIPLE: GAUSSIAN FREE PARTICLE

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 3.19.

As another example of the energy-time uncertainty relation, we can look again at the example of a travelling free particle with a Gaussian wave packet. We have already worked out most of what we need to test the uncertainty relation:

$$(1) \quad \langle x \rangle = \frac{l\hbar t}{m}$$

$$(2) \quad \langle x^2 \rangle = \frac{1 + (2a\hbar t/m)^2 + a(2\hbar l t/m)^2}{4a}$$

$$(3) \quad \langle p^2 \rangle = \hbar^2(a + l^2)$$

From this we get

$$(4) \quad \langle H \rangle = \frac{\langle p^2 \rangle}{2m}$$

$$(5) \quad = \frac{\hbar^2(a + l^2)}{2m}$$

We still need $\langle H^2 \rangle$. From our previous calculations, we have the wave function:

$$(6) \quad \Psi(x, t) = \left(\frac{2a}{\pi}\right)^{1/4} \frac{e^{(-ax^2 + i(lx - \hbar l^2 t/2m))/(1 + 2i\hbar a t/m)}}{\sqrt{1 + 2i\hbar a t/m}}$$

We can calculate $\langle H^2 \rangle$ by direct integration, using Maple:

$$(7) \quad \langle H^2 \rangle = \frac{\hbar^4}{4m^2} \int_{-\infty}^{\infty} \left| \frac{d^2\Psi(x, t)}{dx^2} \right|^2 dx$$

$$(8) \quad = \frac{\hbar^4}{4m^2} (3a^2 + 6al^2 + l^4)$$

As a check on this result, we can work out the units (always a good test to make sure you haven't dropped a factor somewhere). From the original wave function, since exponents must be dimensionless, we know that a has dimensions $distance^{-2}$ and l has $distance^{-1}$. Planck's constant has dimensions of $energy \times time$, so the expression above has overall units of $energy^4 \times time^4 \times mass^{-2} \times distance^{-4} = energy^2$. (Recall kinetic energy is $mv^2/2$.) It's also worth noting that $\langle H^2 \rangle$ is independent of time.

From here, we can get σ_H^2 :

$$(9) \quad \sigma_H^2 = \langle H^2 \rangle - \langle H \rangle^2$$

$$(10) \quad = \frac{a\hbar^4}{2m^2} (a + 2l^2)$$

We also have, from above

$$(11) \quad \sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2$$

$$(12) \quad = \frac{m^2 + 4(at\hbar)^2}{4am^2}$$

We're trying to show that $\sigma_H \sigma_x \geq \frac{\hbar}{2} \frac{d\langle x \rangle}{dt} = l\hbar^2/2m$ from above. So we want

$$(13) \quad \sigma_H^2 \sigma_x^2 \geq \frac{l^2 \hbar^4}{4m^2}$$

$$(14) \quad \frac{a\hbar^4}{2m^2} (a + 2l^2) \frac{m^2 + 4(at\hbar)^2}{4am^2} \geq \frac{l^2 \hbar^4}{4m^2}$$

The minimum of the LHS occurs at $t = 0$ so if the inequality is true there, it is true always. In this case, it reduces to

$$(15) \quad a + 2l^2 \geq 2l^2$$

$$(16) \quad a \geq 0$$

This final condition is certainly true (it is required for the Gaussian wave form to converge at large x), so the uncertainty condition is verified.