

SPECTRAL DECOMPOSITION OF OPERATORS

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Section 3.6; Problem 3.24.

Suppose we have an operator \hat{Q} with a complete, orthonormal set of eigenvectors, so that

$$(1) \quad \hat{Q}|e_m\rangle = q_m|e_m\rangle$$

where q_m is the eigenvalue. We can write the operator as a *spectral decomposition* operator, as follows.

Since the eigenvectors of \hat{Q} form a complete, orthonormal set, we can write any vector $|\alpha\rangle$ in terms of that set:

$$(2) \quad |\alpha\rangle = \sum_m a_m|e_m\rangle$$

where a_m is the coefficient of the basis vector $|e_m\rangle$. Applying \hat{Q} directly to this expansion gives:

$$(3) \quad \hat{Q}|\alpha\rangle = \sum_m a_m q_m |e_m\rangle$$

However, if we write \hat{Q} as

$$(4) \quad \hat{Q} = \sum_n q_n |e_n\rangle\langle e_n|$$

we get

$$(5) \quad \hat{Q}|\alpha\rangle = \left[\sum_n q_n |e_n\rangle\langle e_n| \right] \sum_m a_m |e_m\rangle$$

$$(6) \quad = \sum_{n,m} q_n a_m \delta_{nm} |e_n\rangle\langle e_m|$$

$$(7) \quad = \sum_m a_m q_m |e_m\rangle$$

where the result follows from the orthonormality of the set of eigenvectors: $\langle e_m | e_n \rangle = \delta_{mn}$. Since this spectral decomposition form of \hat{Q} has the same action on any vector as its original form, the two forms must be equivalent.