

## LEGENDRE POLYNOMIALS: GENERATION BY GRAM-SCHMIDT PROCESS

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 3.25.

Starting with the functions  $1, x, x^2, x^3$  we can apply the Gram-Schmidt orthogonalization procedure to generate some polynomials that are orthonormal on the interval  $x \in [-1, 1]$ . (In this post, we use  $f_1, f_2$  etc for the original vectors and primed terms  $f'_1, f'_2$  etc for the orthonormal vectors obtained after Gram-Schmidt. This is different from the earlier post where we used a subscript 1 for the original and subscript 2 for the orthonormal vectors. Apologies for the inconsistency.)

The first function is

$$(1) \quad f_1 = 1$$

To normalize a function over an interval we divide it by the square root of the integral of its square modulus over that interval so here we divide  $f_1$  by  $\sqrt{\int_{-1}^1 1^2 \cdot dx} = \sqrt{2}$ :

$$(2) \quad f'_1 = \frac{1}{\sqrt{2}}$$

Over a symmetric interval, the functions 1 and  $x$  are already orthogonal (1 is even and  $x$  is odd), so we can normalize  $f_2 = x$  directly to get

$$(3) \quad f'_2 = \sqrt{\frac{3}{2}}x$$

To get  $f_3$  we begin by noting that  $x^2$  and  $x$  are orthogonal (even versus odd again), so the Gram-Schmidt process reduces to

$$(4) \quad f_3 = x^2 - \langle f'_1 | x^2 \rangle |f'_1\rangle$$

$$(5) \quad = x^2 - \frac{1}{3}$$

Normalizing this gives

$$(6) \quad f'_3 = \sqrt{\frac{5}{8}}(3x^2 - 1)$$

Finally, noting that  $x^3$  is orthogonal to  $x^2$  and constants, we have

$$(7) \quad f_4 = x^3 - \langle f'_2 | x^3 \rangle |f'_2\rangle$$

$$(8) \quad = x^3 - \frac{3}{5}x$$

Normalizing gives

$$(9) \quad f'_4 = \sqrt{\frac{7}{8}}(5x^3 - 3x)$$

Apart from the normalization, these orthonormalized polynomials are the same as the Legendre polynomials.