LEGENDRE POLYNOMIALS: GENERATION BY GRAM-SCHMIDT PROCESS

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 3.25.

Starting with the functions $1, x, x^2, x^3$ we can apply the Gram-Schmidt orthogonalization procedure to generate some polynomials that are orthonormal on the interval $x \in [-1, 1]$. (In this post, we use f_1, f_2 etc for the original vectors and primed terms f_1', f_2' etc for the orthonormal vectors obtained after Gram-Schmidt. This is different from the earlier post where we used a subscript 1 for the original and subscript 2 for the orthonormal vectors. Apologies for the inconsistency.)

The first function is

$$f_1 = 1 \tag{0.1}$$

To normalize a function over an interval we divide it by the square root of the integral of its square modulus over that interval so here we divide f_1 by $\sqrt{\int_{-1}^{1} 1^2 \cdot dx} = \sqrt{2}$:

$$f_1' = \frac{1}{\sqrt{2}} \tag{0.2}$$

Over a symmetric interval, the functions 1 and x are already orthogonal (1 is even and x is odd), so we can normalize $f_2 = x$ directly to get

$$f_2' = \sqrt{\frac{3}{2}} x \tag{0.3}$$

To get f_3 we begin by noting that x^2 and x are orthogonal (even versus odd again), so the Gram-Schmidt process reduces to

$$\begin{aligned} f_3 &= x^2 - \langle f_1' | x^2 \rangle | f_1' \rangle && (0.4) \\ &= x^2 - \frac{1}{3} && (0.5) \end{aligned}$$

Normalizing this gives

$$f_3' = \sqrt{\frac{5}{8}}(3x^2 - 1) \tag{0.6}$$

Finally, noting that x^3 is orthogonal to x^2 and constants, we have

$$f_4 = x^3 - \langle f_2' | x^3 \rangle | f_2' \rangle \tag{0.7}$$

$$= x^3 - \frac{3}{5}x \tag{0.8}$$

Normalizing gives

$$f_4' = \sqrt{\frac{7}{8}}(5x^3 - 3x) \tag{0.9}$$

Apart from the normalization, these orthonormalized polynomials are the same as the Legendre polynomials.