LEGENDRE POLYNOMIALS: GENERATION BY GRAM-SCHMIDT PROCESS

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 3.25.

Starting with the functions $1,x,x^2,x^3$ we can apply the Gram-Schmidt orthogonalization procedure to generate some polynomials that are orthonormal on the interval $x\in[-1,1]$. (In this post, we use f_1,f_2 etc for the original vectors and primed terms f_1',f_2' etc for the orthonormal vectors obtained after Gram-Schmidt. This is different from the earlier post where we used a subscript 1 for the original and subscript 2 for the orthonormal vectors. Apologies for the inconsistency.)

The first function is

$$f_1 = 1 \tag{1}$$

To normalize a function over an interval we divide it by the square root of the integral of its square modulus over that interval so here we divide f_1 by $\sqrt{\int_{-1}^{1}1^2\cdot dx}=\sqrt{2}$:

$$f_1' = \frac{1}{\sqrt{2}} \tag{2}$$

Over a symmetric interval, the functions 1 and x are already orthogonal (1 is even and x is odd), so we can normalize $f_2=x$ directly to get

$$f_2' = \sqrt{\frac{3}{2}}x \tag{3}$$

To get f_3 we begin by noting that x^2 and x are orthogonal (even versus odd again), so the Gram-Schmidt process reduces to

$$\begin{aligned} f_3 &= x^2-\langle f_1'|x^2\rangle|f_1'\rangle && (4)\\ &= x^2-\frac{1}{3} && (5)\end{aligned}$$

Normalizing this gives

$$f_3' = \sqrt{\frac{5}{8}}(3x^2 - 1) \tag{6}$$

Finally, noting that x^3 is orthogonal to x^2 and constants, we have

$$\begin{aligned} f_4 &= x^3 - \langle f_2' | x^3 \rangle | f_2' \rangle & (7) \\ &= x^3 - \frac{3}{5}x & (8) \end{aligned}$$

Normalizing gives

$$f_4' = \sqrt{\frac{7}{8}}(5x^3 - 3x) \tag{9}$$

Apart from the normalization, these orthonormalized polynomials are the same as the Legendre polynomials.