ANTI-HERMITIAN OPERATORS

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 3.26.

A hermitian operator is equal to its hermitian conjugate (which, remember, is the complex conjugate of the transpose of the matrix representing the operator). That is,

$$\hat{Q}^{\dagger}=\hat{Q} \tag{1}$$

This has the consequence that for inner products

$$\begin{aligned}\langle f|\hat{Q}g\rangle &= \langle\hat{Q}^{\dagger}f|g\rangle &(2)\\ &= \langle\hat{Q}f|g\rangle &(3)\end{aligned}$$

An *anti-hermitian* operator is equal to the negative of its hermitian conjugate, that is

$$\hat{Q}^{\dagger}=-\hat{Q} \tag{4}$$

In inner products, this means

$$\begin{aligned}\langle f|\hat{Q}g\rangle &= \langle\hat{Q}^{\dagger}f|g\rangle &(5)\\ &= -\langle\hat{Q}f|g\rangle &(6)\end{aligned}$$

The expectation value of an anti-hermitian operator is:

$$\begin{aligned}\langle f|\hat{Q}f\rangle &= \langle\hat{Q}^{\dagger}f|f\rangle &(7)\\ &= -\langle\hat{Q}f|f\rangle &(8)\\ &= -\langle Q\rangle^{*} &(9)\end{aligned}$$

But $\langle f|\hat{Q}f\rangle=\langle Q\rangle$ so $\langle Q\rangle=-\langle Q\rangle^{*}$, which means the expectation value must be pure imaginary.

For two hermitian operators $\hat{Q}$ and $\hat{R}$ we have

$$\begin{aligned}
\left[\hat{Q},\hat{R}\right] &= \hat{Q}\hat{R}-\hat{R}\hat{Q} &(10)\\
[\hat{Q},\hat{R}]^{\dagger} &= \hat{R}^{\dagger}\hat{Q}^{\dagger}-\hat{Q}^{\dagger}\hat{R}^{\dagger} &(11)\\
&= \hat{R}\hat{Q}-\hat{Q}\hat{R} &(12)\\
&= \left[\hat{R},\hat{Q}\right] &(13)\\
&= -[\hat{Q},\hat{R}] &(14)
\end{aligned}$$

where we have used the hermitian property $\hat{Q}^{\dagger}=\hat{Q}$ to get the third line. Thus the commutator of two hermitian operators is anti-hermitian.

If two operators $\hat{S}$ and $\hat{T}$ are anti-hermitian, a similar derivation shows that $[\hat{S},\hat{T}]^{\dagger}=-[\hat{S},\hat{T}]$ also:

$$\begin{aligned}
\left[\hat{S},\hat{T}\right] &= \hat{S}\hat{T}-\hat{T}\hat{S} &(15)\\
[\hat{S},\hat{T}]^{\dagger} &= \hat{T}^{\dagger}\hat{S}^{\dagger}-\hat{S}^{\dagger}\hat{T}^{\dagger} &(16)\\
&= (-\hat{T})(-\hat{S})-(-\hat{S})(-\hat{T}) &(17)\\
&= -[\hat{S},\hat{T}] &(18)
\end{aligned}$$

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