SEQUENTIAL MEASUREMENTS

Suppose we have a two-state system, and there are two observables, $A$ and $B$, that can be measured. Observable $A$ has two eigenstates (eigenvectors, if you like) called $\psi_1$ and $\psi_2$ with eigenvalues (possible values that can be observed) of $a_1$ and $a_2$ respectively. For $B$, the eigenstates are $\phi_1$ and $\phi_2$, with eigenvalues $b_1$ and $b_2$.

Since both observables have a complete set of eigenstates, the state of the system can be expressed in terms of either set (in a way similar to that in which we can express states in either position or momentum space when we’re considering one-dimensional states). In particular, we can express the eigenstates of one observable in terms of the eigenstates of the other. For example, we might have

\begin{align}
\psi_1 &= \frac{1}{5}(3\phi_1 + 4\phi_2) \\
\psi_2 &= \frac{1}{5}(4\phi_1 - 3\phi_2)
\end{align}

Now suppose that observable $A$ is measured and the value is found to be $a_1$. If a measurement gives a particular eigenvalue, then the state of the system collapses to the eigenfunction for that eigenvalue, so the state of the system immediately after the measurement is $\psi_1$.

If we now measure $B$, we can find the probability of getting either $b_1$ or $b_2$. Since $\psi_1 = \frac{1}{5}(3\phi_1 + 4\phi_2)$, the probability of getting $b_1$ is $9/25$ and of $b_2$ is $16/25$.

After measuring $B$, we measure $A$ again. What is the probability of again getting $a_1$? The important point here is that we did measure $B$ before attempting to measure $A$ again. If we hadn’t measured $B$, then the system would still be in state $\psi_1$, which means we would get $a_1$ with 100% certainty. However, measuring $B$ will force the system into one of the eigenstates of $B$. If we don’t know the outcome of the measurement of $B$, we must use conditional probabilities to calculate the probability of measuring $a_1$ after measuring $B$. This is
\[ Prob(a_1) = Prob(a_1|b_1)Prob(b_1) + Prob(a_1|b_2)Prob(b_2) \] (3)

where \( Prob(a_1|b_1) \) means 'probability of \( a_1 \) given \( b_1 \).

To get the conditional probabilities, we can invert the equations relating the two sets of eigenfunctions to get

\[ \phi_1 = \frac{1}{5}(3\psi_1 + 4\psi_2) \] (4)
\[ \phi_2 = \frac{1}{5}(4\psi_1 - 3\psi_2) \] (5)

From here we can see that

\[ Prob(a_1|b_1) = \frac{9}{25} \] (6)
\[ Prob(a_1|b_2) = \frac{16}{25} \] (7)

so, using the probabilities above for getting each measurement of \( B \) prior to the final measurement of \( A \), the final probability of getting \( a_1 \) after measuring \( B \) is \((9/25)^2 + (16/25)^2 = \frac{337}{625} = 0.5392\).