INFINITE SQUARE WELL: MOMENTUM SPACE WAVE FUNCTIONS

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 3.28.

We've calculated the momentum space wave function for the ground state of the harmonic oscillator, and we can use the same technique to investigate the infinite square well. We have:

$$\begin{aligned}
\Phi_n(p,t) &= \frac{1}{\sqrt{2\pi h}}\int_0^a e^{-ipx/\hbar}\Psi_n(x,t)dx && (1)\\
&= \frac{1}{\sqrt{2\pi h}}\int_0^a e^{-ipx/\hbar}\sin\left(\frac{n\pi x}{a}\right)e^{-i(n^2\pi^2\hbar/2ma^2)t}dx && (2)\\
&= \frac{n\hbar^{3/2}\sqrt{\pi a}}{p^2a^2-n^2\pi^2\hbar^2}((-1)^n e^{ipa/\hbar}-1)e^{-i(n^2\pi^2\hbar/2ma^2)t} && (3)
\end{aligned}$$

The square modulus of this function is

$$|\Phi_n(p,t)|^2 = 2\pi n^2\hbar^3 a\frac{1+(-1)^{n+1}\cos(ap/\hbar)}{(p^2a^2-n^2\pi^2\hbar^2)^2} \tag{4}$$

We can write this in terms of the auxiliary variable $\rho \equiv pa/\hbar$:

$$|\Phi_n(p,t)|^2 = \frac{2\pi n^2 a}{\hbar}\left[\frac{1+(-1)^{n+1}\cos\rho}{(\rho^2-n^2\pi^2)^2}\right] \tag{5}$$

To investigate the behaviour of this formula around the points $\rho = n\pi$, we can write it as

$$|\Phi_n(p,t)|^2 = \frac{2\pi n^2 a}{\hbar}\left[\frac{1+(-1)^{n+1}\cos\rho}{(\rho+n\pi)^2(\rho-n\pi)^2}\right] \tag{6}$$

We can now expand the numerator in a Taylor series about the point $\rho = n\pi$:

$$1+(-1)^{n+1}\cos\rho = 1+(-1)^{2n+1}-\frac{(-1)^{2n+1}}{2}(\rho-n\pi)^2+\frac{(-1)^{2n+1}}{24}(\rho-n\pi)^4+\ldots \tag{7}$$

$$=\frac{1}{2}(\rho-n\pi)^2-\frac{1}{24}(\rho-n\pi)^4+\ldots \tag{8}$$

where we've used the fact that $(-1)^{2n+1}=-1$ for all n, since the exponent is always odd.

Dividing by the denominator, we get

$$|\Phi_n(p,t)|^2=\frac{2\pi n^2 a}{\hbar(\rho+n\pi)^2}\left(\frac{1}{2}-\frac{1}{24}(\rho-n\pi)^2+\ldots\right) \tag{9}$$

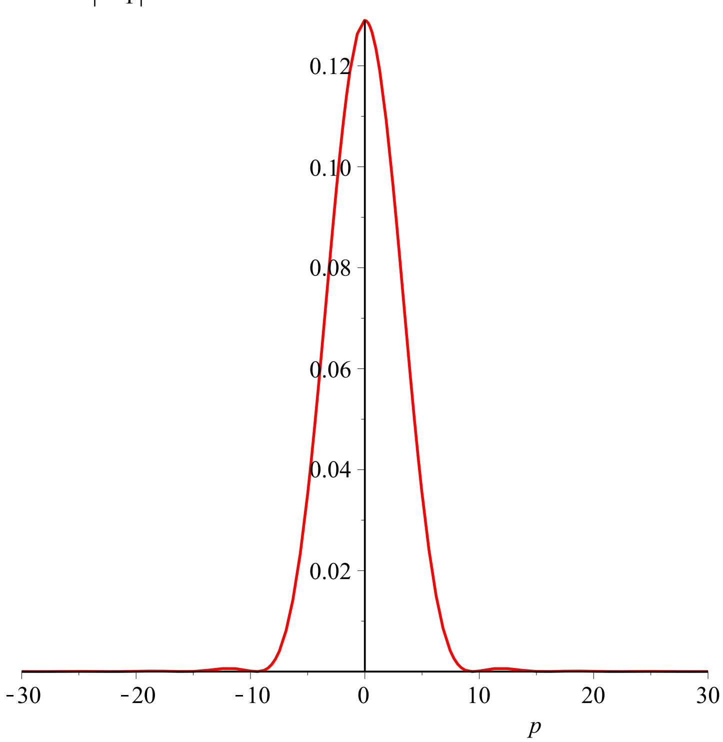
If we now take the limit, we get

$$\lim_{\rho\to n\pi}|\Phi_n(p,t)|^2=\frac{a}{4\pi\hbar} \tag{10}$$

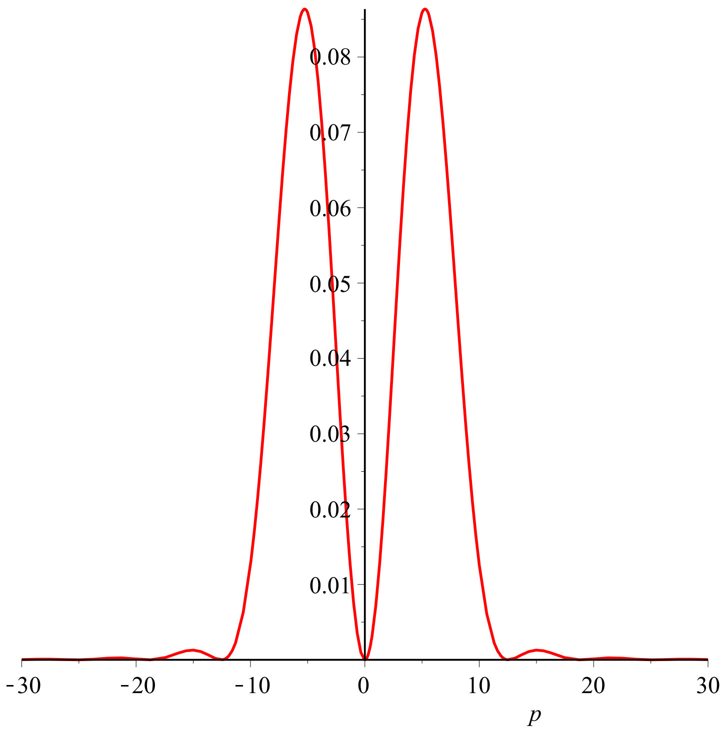
The limit is independent of n.

The plots for $|\Phi_1|^2$ and $|\Phi_2|^2$ are shown below.

Here is $|\Phi_1|^2$:



Here is $|\Phi_2|^2$:



By direct calculation using 4 (and Maple), we can get

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} p^2 |\Phi_n(p,t)|^2 dp \quad (11)$$

$$= (n\pi\hbar/a)^2 \quad (12)$$

which is the same as that obtained by doing the calculation in position space. For good measure, we can also calculate $\langle p \rangle = 0$, either by direct calculation or by observing that since $|\Phi_n(p,t)|^2$ is even for all n, $\langle p \rangle$ is always the integral of an odd function over a symmetric interval, so is always zero.