

MOMENTUM SPACE REPRESENTATION OF FINITE WAVE FUNCTION

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 3.29.

Suppose we have a wave function that is purely sinusoidal in position space, but is defined only over a finite interval, so we have

$$(0.1) \quad \Psi(x, 0) = \begin{cases} \frac{1}{\sqrt{2n\lambda}} e^{i2\pi x/\lambda} & -n\lambda < x < n\lambda \\ 0 & \text{otherwise} \end{cases}$$

where n is a positive integer and λ is a positive constant.

We can calculate the momentum space wave function in the usual way:

$$(0.2) \quad \Phi_n(p, 0) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-n\lambda}^{n\lambda} e^{-ipx/\hbar} \Psi_n(x, 0) dx$$

$$(0.3) \quad = \frac{1}{\sqrt{2\pi\hbar}} \int_{-n\lambda}^{n\lambda} e^{-ipx/\hbar} \frac{1}{2n\lambda} e^{i2\pi x/\lambda} dx$$

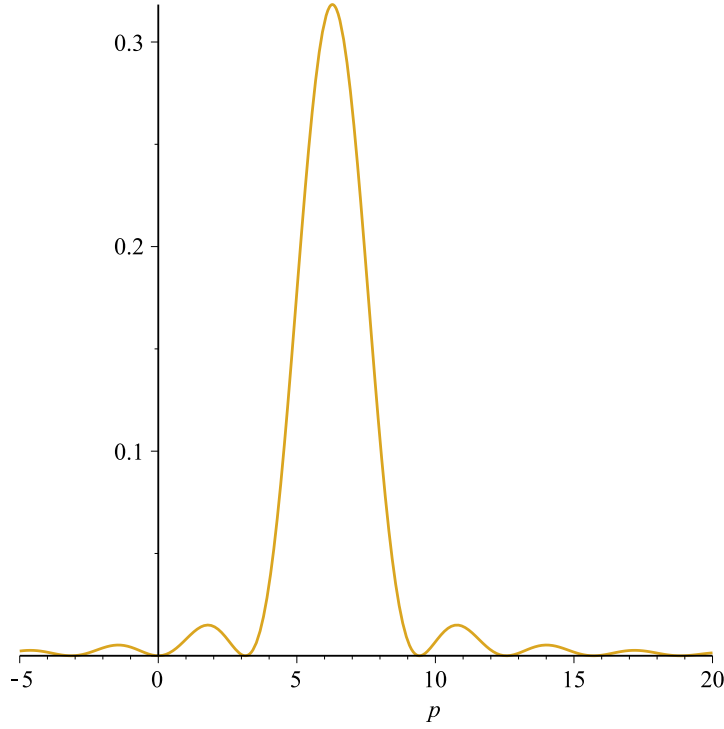
$$(0.4) \quad = \sqrt{\frac{\lambda\hbar}{\pi n}} \frac{\sin(np\lambda/\hbar)}{(p\lambda - 2\pi\hbar)}$$

The plot of $|\Psi_n(x, 0)|^2$ is constant with value $1/2n\lambda$ in the interval $[-n\lambda, n\lambda]$.

To plot $|\Phi_1(p, 0)|^2$ we can define the auxiliary variable $\rho \equiv p\lambda/\hbar$ so we have

$$(0.5) \quad |\Phi_1(p, 0)|^2 = \frac{\lambda}{\pi\hbar} \left[\frac{\sin^2(\rho)}{(\rho - 2\pi)^2} \right]$$

We get the following plot:



The peak occurs at $\rho = 2\pi$, or $p = 2\pi\hbar/\lambda$. The first zero on either side of the peak occurs at $\rho = 2\pi \pm \pi = \pi, 3\pi$, so the width of the central peak is $w_p = 2\pi\hbar/\lambda$. The width of the position space curve is simply $w_x = 2\lambda$. If we take these two widths as measures of the uncertainties in position and momentum, we get $w_x w_p = 4\pi\hbar$.

For general n , the two widths are $w_x = 2n\lambda$ and $w_p = 2\pi\hbar/n\lambda$. As $n \rightarrow \infty$, the width in x -space gets larger, while that in momentum space gets smaller, eventually reaching the point where there is no uncertainty in momentum but total uncertainty in position. The product $w_x w_p = 4\pi\hbar$ is independent of n and is well above the uncertainty principle limit of $\hbar/2$.

We can attempt to calculate $\langle p \rangle$ and $\langle p^2 \rangle$ in the usual way using integration. We get

$$(0.6) \quad \langle p \rangle = \int_{-\infty}^{\infty} p |\Phi_1(p, 0)|^2 dp$$

Converting to the variable ρ we get

$$(0.7) \quad \langle p \rangle = \frac{\hbar}{\pi\lambda} \int_{-\infty}^{\infty} \rho \frac{\sin^2(\rho)}{(\rho - 2\pi)^2} d\rho$$

Maple has a problem with the infinite limits, but if we try it for finite limits which are centred on $\rho = 2\pi$, we find

$$(0.8) \quad \frac{\hbar}{\pi\lambda} \int_{2\pi-a}^{2\pi+a} \rho \frac{\sin^2(\rho)}{(\rho - 2\pi)^2} d\rho = \frac{4\hbar}{\lambda a} (\cos^2 a + a\text{Si}(2a) - 1)$$

Here $\text{Si}(x)$ is the *sine integral*, defined as

$$(0.9) \quad \text{Si}(x) \equiv \int_0^x \frac{\sin t}{t} dt$$

The important property of the sine integral is

$$(0.10) \quad \lim_{x \rightarrow \infty} \text{Si}(x) = \frac{\pi}{2}$$

so we can now take the limit of the above integral when $a \rightarrow \infty$ and get

$$(0.11) \quad \langle p \rangle = \lim_{a \rightarrow \infty} \frac{4\hbar}{\lambda a} (\cos^2 a + a\text{Si}(2a) - 1)$$

$$(0.12) \quad = \frac{2\pi\hbar}{\lambda}$$

This result isn't much of a surprise, since it says the mean momentum is just that at the peak in the graph above.

Attempting to calculate $\langle p^2 \rangle$ requires integrating the function $p^2 |\Phi_n(p, 0)|^2$ over an infinite range. This involves the product of a sine-squared function with the term $p^2 / (p\lambda - 2\pi\hbar)^2 = 1 / (\lambda - 2\pi\hbar/p)^2$. As $p \rightarrow \pm\infty$, this second term tends to a constant ($1/\lambda$), so the integrand tends to the sine-squared term (which is always non-negative), so the integral diverges to infinity. This probably results from the fact that $\Psi(x, 0)$ has two step functions, and since the momentum operator involves the derivative of position, this will produce a delta function at the two end points. Finding $\langle p^2 \rangle$ thus involves integrating the square of a delta function and I wouldn't want to speculate on what that would produce.