MOMENTUM SPACE REPRESENTATION OF FINITE WAVE FUNCTION

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 3.29.

Suppose we have a wave function that is purely sinusoidal in position space, but is defined only over a finite interval, so we have

$$\Psi(x,0)=\begin{cases}\frac{1}{\sqrt{2n\lambda}}e^{i2\pi x/\lambda} & -n\lambda<x<n\lambda\\ 0 & \text{otherwise}\end{cases} \tag{1}$$

where n is a positive integer and λ is a positive constant.

We can calculate the momentum space wave function in the usual way:

$$\Phi_n(p,0) = \frac{1}{\sqrt{2\pi h}}\int_{-n\lambda}^{n\lambda}e^{-ipx/\hbar}\Psi_n(x,0)dx \tag{2}$$

$$= \frac{1}{\sqrt{2\pi h}}\int_{-n\lambda}^{n\lambda}e^{-ipx/\hbar}\frac{1}{2n\lambda}e^{i2\pi x/\lambda}dx \tag{3}$$

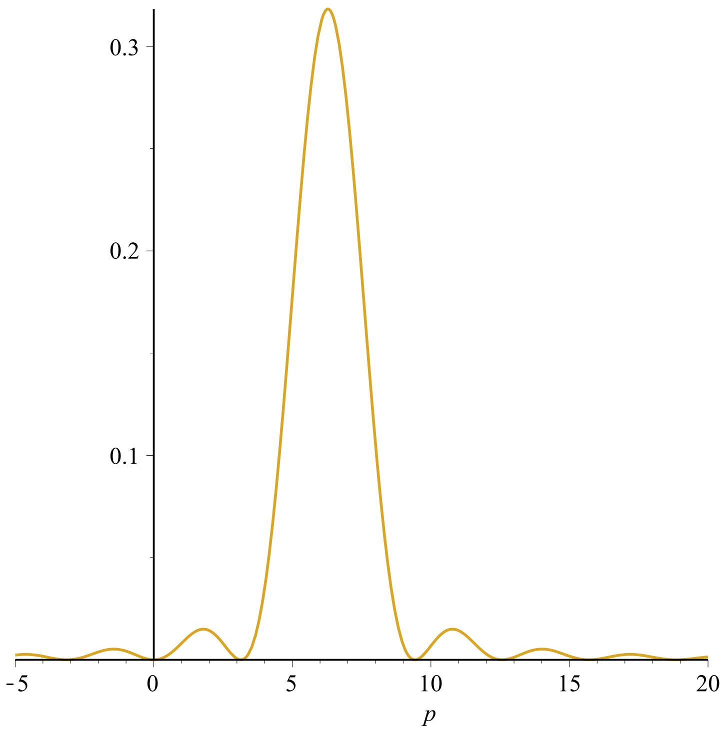
$$= \sqrt{\frac{\lambda\hbar}{\pi n}}\frac{\sin(np\lambda/\hbar)}{(p\lambda-2\pi\hbar)} \tag{4}$$

The plot of $|\Psi_n(x,0)|^2$ is constant with value $1/2n\lambda$ in the interval $[-n\lambda,n\lambda]$.

To plot $|\Phi_1(p,0)|^2$ we can define the auxiliary variable $\rho\equiv p\lambda/\hbar$ so we have

$$|\Phi_1(p,0)|^2=\frac{\lambda}{\pi\hbar}\left[\frac{\sin^2(\rho)}{(\rho-2\pi)^2}\right] \tag{5}$$

We get the following plot:



The peak occurs at $\rho = 2\pi$, or $p = 2\pi\hbar/\lambda$. The first zero on either side of the peak occurs at $\rho = 2\pi \pm \pi = \pi, 3\pi$, so the width of the central peak is $w_p = 2\pi\hbar/\lambda$. The width of the position space curve is simply $w_x = 2\lambda$. If we take these two width as measures of the uncertainties in position and momentum, we get $w_x w_p = 4\pi\hbar$.

For general n, the two widths are $w_x = 2n\lambda$ and $w_p = 2\pi\hbar/n\lambda$. As $n \to \infty$, the width in x-space gets larger, while that in momentum space gets smaller, eventually reaching the point where there is no uncertainty in momentum but total uncertainty in position. The product $w_x w_p = 4\pi\hbar$ is independent of n and is well above the uncertainty principle limit of $\hbar/2$.

We can attempt to calculate $\langle p \rangle$ and $\langle p^2 \rangle$ in the usual way using integration. We get

$$\langle p \rangle = \int_{-\infty}^{\infty} p |\Phi_1(p,0)|^2 dp \tag{6}$$

Converting to the variable ρ we get

$$\langle p \rangle = \frac{\hbar}{\pi\lambda} \int_{-\infty}^{\infty} \rho \frac{\sin^2(\rho)}{(\rho - 2\pi)^2} d\rho \tag{7}$$

Maple has a problem with the infinite limits, but if we try it for finite limits which are centred on $\rho = 2\pi$, we find

$$\frac{\hbar}{\pi\lambda}\int_{2\pi-a}^{2\pi+a}\rho\frac{\sin^{2}(\rho)}{(\rho-2\pi)^{2}}d\rho=\frac{4\hbar}{\lambda a}\left(\cos^{2}a+a\mathrm{Si}(2a)-1\right) \tag{8}$$

Here $\mathrm{Si}(x)$ is the *sine integral*, defined as

$$\mathrm{Si}(x)\equiv\int_{0}^{x}\frac{\sin t}{t}dt \tag{9}$$

The important property of the sine integral is

$$\lim_{x\to\infty}\mathrm{Si}(x)=\frac{\pi}{2} \tag{10}$$

so we can now take the limit of the above integral when $a\to\infty$ and get

$$\langle p\rangle = \lim_{a\to\infty}\frac{4\hbar}{\lambda a}\left(\cos^{2}a+a\mathrm{Si}(2a)-1\right) \tag{11}$$

$$= \frac{2\pi\hbar}{\lambda} \tag{12}$$

This result isn't much of a surprise, since it says the mean momentum is just that at the peak in the graph above.

Attempting to calculate $\langle p^2\rangle$ requires integrating the function $p^2|\Phi_n(p,0)|^2$ over an infinite range. This involves the product of a sine-squared function with the term $p^2/(p\lambda-2\pi\hbar)^2=1/(\lambda-2\pi\hbar/p)^2$. As $p\to\pm\infty$, this second term tends to a constant $(1/\lambda)$, so the integrand tends to the sine-squared term (which is always non-negative), so the integral diverges to infinity. This probably results from the fact that $\Psi(x,0)$ has two step functions, and since the momentum operator involves the derivative of position, this will produce a delta function at the two end points. Finding $\left\langle p^2\right\rangle$ thus involves integrating the square of a delta function and I wouldn't want to speculate on what that would produce.