

## MOMENTUM SPACE REPRESENTATION OF FINITE WAVE FUNCTION

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 3.29.

Suppose we have a wave function that is purely sinusoidal in position space, but is defined only over a finite interval, so we have

$$\Psi(x, 0) = \begin{cases} \frac{1}{\sqrt{2n\lambda}} e^{i2\pi x/\lambda} & -n\lambda < x < n\lambda \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

where  $n$  is a positive integer and  $\lambda$  is a positive constant.

We can calculate the momentum space wave function in the usual way:

$$\Phi_n(p, 0) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-n\lambda}^{n\lambda} e^{-ipx/\hbar} \Psi_n(x, 0) dx \quad (2)$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-n\lambda}^{n\lambda} e^{-ipx/\hbar} \frac{1}{2n\lambda} e^{i2\pi x/\lambda} dx \quad (3)$$

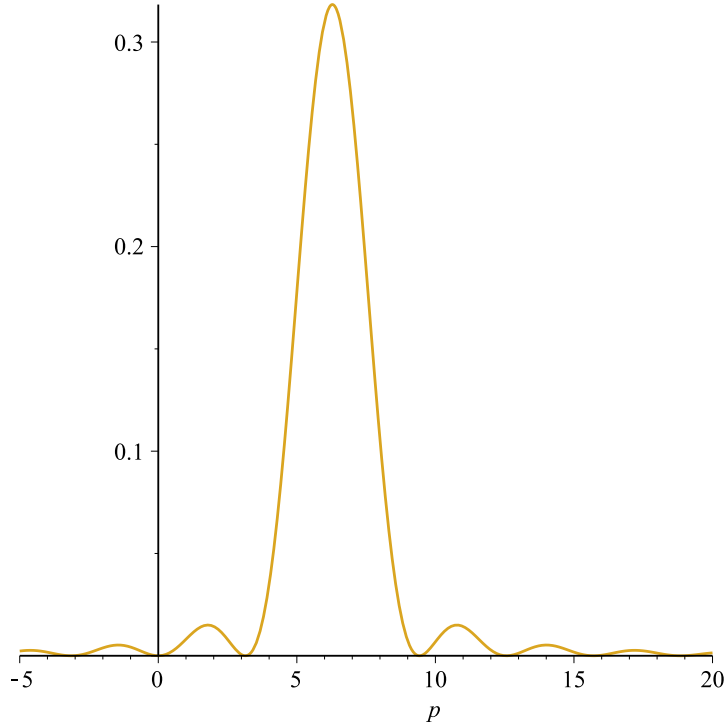
$$= \sqrt{\frac{\lambda\hbar}{\pi n}} \frac{\sin(np\lambda/\hbar)}{(p\lambda - 2\pi\hbar)} \quad (4)$$

The plot of  $|\Psi_n(x, 0)|^2$  is constant with value  $1/2n\lambda$  in the interval  $[-n\lambda, n\lambda]$ .

To plot  $|\Phi_1(p, 0)|^2$  we can define the auxiliary variable  $\rho \equiv p\lambda/\hbar$  so we have

$$|\Phi_1(p, 0)|^2 = \frac{\lambda}{\pi\hbar} \left[ \frac{\sin^2(\rho)}{(\rho - 2\pi)^2} \right] \quad (5)$$

We get the following plot:



The peak occurs at  $\rho = 2\pi$ , or  $p = 2\pi\hbar/\lambda$ . The first zero on either side of the peak occurs at  $\rho = 2\pi \pm \pi = \pi, 3\pi$ , so the width of the central peak is  $w_p = 2\pi\hbar/\lambda$ . The width of the position space curve is simply  $w_x = 2\lambda$ . If we take these two widths as measures of the uncertainties in position and momentum, we get  $w_x w_p = 4\pi\hbar$ .

For general  $n$ , the two widths are  $w_x = 2n\lambda$  and  $w_p = 2\pi\hbar/n\lambda$ . As  $n \rightarrow \infty$ , the width in  $x$ -space gets larger, while that in momentum space gets smaller, eventually reaching the point where there is no uncertainty in momentum but total uncertainty in position. The product  $w_x w_p = 4\pi\hbar$  is independent of  $n$  and is well above the uncertainty principle limit of  $\hbar/2$ .

We can attempt to calculate  $\langle p \rangle$  and  $\langle p^2 \rangle$  in the usual way using integration. We get

$$\langle p \rangle = \int_{-\infty}^{\infty} p |\Phi_1(p, 0)|^2 dp \quad (6)$$

Converting to the variable  $\rho$  we get

$$\langle p \rangle = \frac{\hbar}{\pi\lambda} \int_{-\infty}^{\infty} \rho \frac{\sin^2(\rho)}{(\rho - 2\pi)^2} d\rho \quad (7)$$

Maple has a problem with the infinite limits, but if we try it for finite limits which are centred on  $\rho = 2\pi$ , we find

$$\frac{\hbar}{\pi\lambda} \int_{2\pi-a}^{2\pi+a} \rho \frac{\sin^2(\rho)}{(\rho-2\pi)^2} d\rho = \frac{4\hbar}{\lambda a} (\cos^2 a + a\text{Si}(2a) - 1) \quad (8)$$

Here  $\text{Si}(x)$  is the *sine integral*, defined as

$$\text{Si}(x) \equiv \int_0^x \frac{\sin t}{t} dt \quad (9)$$

The important property of the sine integral is

$$\lim_{x \rightarrow \infty} \text{Si}(x) = \frac{\pi}{2} \quad (10)$$

so we can now take the limit of the above integral when  $a \rightarrow \infty$  and get

$$\langle p \rangle = \lim_{a \rightarrow \infty} \frac{4\hbar}{\lambda a} (\cos^2 a + a\text{Si}(2a) - 1) \quad (11)$$

$$= \frac{2\pi\hbar}{\lambda} \quad (12)$$

This result isn't much of a surprise, since it says the mean momentum is just that at the peak in the graph above.

Attempting to calculate  $\langle p^2 \rangle$  requires integrating the function  $p^2 |\Phi_n(p, 0)|^2$  over an infinite range. This involves the product of a sine-squared function with the term  $p^2 / (p\lambda - 2\pi\hbar)^2 = 1 / (\lambda - 2\pi\hbar/p)^2$ . As  $p \rightarrow \pm\infty$ , this second term tends to a constant ( $1/\lambda$ ), so the integrand tends to the sine-squared term (which is always non-negative), so the integral diverges to infinity. This probably results from the fact that  $\Psi(x, 0)$  has two step functions, and since the momentum operator involves the derivative of position, this will produce a delta function at the two end points. Finding  $\langle p^2 \rangle$  thus involves integrating the square of a delta function and I wouldn't want to speculate on what that would produce.