

MOMENTUM SPACE: ANOTHER EXAMPLE

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 3.30.

Another example of calculations in momentum space. Suppose we have the initial state of a wave function in position space given by

$$(0.1) \quad \Psi(x, 0) = \frac{A}{x^2 + a^2}$$

where a and A are constants.

(a) Applying normalization

$$(0.2) \quad 1 = A^2 \int_{-\infty}^{\infty} \left(\frac{1}{x^2 + a^2} \right)^2 dx$$

$$(0.3) \quad = \frac{\pi A^2}{2a^3}$$

$$(0.4) \quad A = \sqrt{\frac{2}{\pi}} a^{3/2}$$

(b) We can use the position space wave function to work out $\langle x \rangle$ and $\langle x^2 \rangle$. Since $|\Psi(x, 0)|^2 = A^2/(x^2 + a^2)^2$ is even, $\langle x \rangle = 0$. For $\langle x^2 \rangle$, we find

$$(0.5) \quad \langle x^2 \rangle = \frac{2}{\pi} a^3 \int_{-\infty}^{\infty} x^2 \left(\frac{1}{x^2 + a^2} \right)^2 dx$$

$$(0.6) \quad = a^2$$

Thus $\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = a$.

(c) To get the momentum space wave function, we must do the integral

$$(0.7) \quad \Phi(p, 0) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \sqrt{\frac{2a^3}{\pi}} \frac{1}{x^2 + a^2} dx$$

Feeding this integral into Maple and telling it to assume p is real gives an answer that depends on the sign of p . Simplifying the expression for the two signs gives the single result:

$$(0.8) \quad \Phi(p, 0) = \sqrt{\frac{a}{\hbar}} e^{-|p|a/\hbar}$$

Note that this is an even function of p , which is correct, since looking at the integral, the complex exponential is the sum of a real, even function (cosine) and an imaginary, odd function (i *sine). The product of this exponential with an even function ($1/(x^2 + a^2)$) integrated over a symmetric interval will be non-zero only for the cosine part, and the cosine itself is even in p , so the result of the integral must also be even in p .

To check the normalization of $\Phi(p, 0)$, we do the integral:

$$(0.9) \quad \int_{-\infty}^{\infty} |\Phi(p, 0)|^2 dp = 2 \frac{a}{\hbar} \int_0^{\infty} e^{-2pa/\hbar} dp$$

$$(0.10) \quad = 1$$

(d) Since $\Phi(p, 0)$ is even, $\langle p \rangle = 0$. For $\langle p^2 \rangle$, we have

$$(0.11) \quad \langle p^2 \rangle = \int_{-\infty}^{\infty} p^2 |\Phi(p, 0)|^2 dp$$

$$(0.12) \quad = 2 \frac{a}{\hbar} \int_0^{\infty} p^2 e^{-2pa/\hbar} dp$$

$$(0.13) \quad = \frac{\hbar^2}{2a^2}$$

Thus $\sigma_p = \hbar/\sqrt{2}a$ and the uncertainty principle is $\sigma_p \sigma_x = \hbar/\sqrt{2}$.