MOMENTUM SPACE: ANOTHER EXAMPLE

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 3.30.

Another example of calculations in momentum space. Suppose we have the initial state of a wave function in position space given by

$$\Psi(x,0)=\frac{A}{x^{2}+a^{2}} \tag{1}$$

where a and A are constants.

(a) Applying normalization

$$\begin{aligned}
1 &= A^{2}\int_{-\infty}^{\infty}\left(\frac{1}{x^{2}+a^{2}}\right)^{2}dx && (2)\\
&= \frac{\pi A^{2}}{2a^{3}} && (3)\\
A &= \sqrt{\frac{2}{\pi}}a^{3/2} && (4)
\end{aligned}$$

(b) We can use the position space wave function to work out $\langle x\rangle$ and $\langle x^{2}\rangle$. Since $|\Psi(x,0)|^{2}=A^{2}/(x^{2}+a^{2})^{2}$ is even, $\langle x\rangle=0$. For $\langle x^{2}\rangle$, we find

$$\begin{aligned}
\langle x^{2}\rangle &= \frac{2}{\pi}a^{3}\int_{-\infty}^{\infty}x^{2}\left(\frac{1}{x^{2}+a^{2}}\right)^{2}dx && (5)\\
&= a^{2} && (6)
\end{aligned}$$

Thus $\sigma_{x}=\sqrt{\langle x^{2}\rangle-\langle x\rangle^{2}}=a$.

(c) To get the momentum space wave function, we must do the integral

$$\Phi(p,0) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}e^{-ipx/\hbar}\sqrt{\frac{2a^{3}}{\pi}}\frac{1}{x^{2}+a^{2}}dx \tag{7}$$

Feeding this integral into Maple and telling it to assume p is real gives an answer that depends on the sign of p. Simplifying the expression for the two signs gives the single result:

$$\Phi(p,0)=\sqrt{\frac{a}{\hbar}}e^{-|p|a/\hbar} \tag{8}$$

Note that this is an even function of p, which is correct, since looking at the integral, the complex exponential is the sum of a real, even function (cosine) and an imaginary, odd function (i*sine). The product of this exponential with an even function $(1/(x^2+a^2))$ integrated over a symmetric interval will be non-zero only for the cosine part, and the cosine itself is even in p, so the result of the integral must also be even in p.

To check the normalization of $\Phi(p,0)$, we do the integral:

$$\int_{-\infty}^{\infty}|\Phi(p,0)|^2dp = 2\frac{a}{\hbar}\int_0^{\infty}e^{-2pa/\hbar}dp \tag{9}$$

$$= 1 \tag{10}$$

(d) Since $\Phi(p,0)$ is even, $\langle p\rangle=0$. For $\langle p^2\rangle$, we have

$$\langle p^2\rangle = \int_{-\infty}^{\infty}p^2|\Phi(p,0)|^2dp \tag{11}$$

$$= 2\frac{a}{\hbar}\int_0^{\infty}p^2e^{-2pa/\hbar}dp \tag{12}$$

$$= \frac{\hbar^2}{2a^2} \tag{13}$$

Thus $\sigma_p=\hbar/\sqrt{2}a$ and the uncertainty principle is $\sigma_p\sigma_x=\hbar/\sqrt{2}$.