

## MOMENTUM SPACE: ANOTHER EXAMPLE

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Post date: 8 Oct 2012.

References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 3.30.

Another example of calculations in momentum space. Suppose we have the initial state of a wave function in position space given by

$$\Psi(x, 0) = \frac{A}{x^2 + a^2} \quad (1)$$

where  $a$  and  $A$  are constants.

(a) Applying normalization

$$1 = A^2 \int_{-\infty}^{\infty} \left( \frac{1}{x^2 + a^2} \right)^2 dx \quad (2)$$

$$= \frac{\pi A^2}{2a^3} \quad (3)$$

$$A = \sqrt{\frac{2}{\pi}} a^{3/2} \quad (4)$$

(b) We can use the position space wave function to work out  $\langle x \rangle$  and  $\langle x^2 \rangle$ . Since  $|\Psi(x, 0)|^2 = A^2/(x^2 + a^2)^2$  is even,  $\langle x \rangle = 0$ . For  $\langle x^2 \rangle$ , we find

$$\langle x^2 \rangle = \frac{2}{\pi} a^3 \int_{-\infty}^{\infty} x^2 \left( \frac{1}{x^2 + a^2} \right)^2 dx \quad (5)$$

$$= a^2 \quad (6)$$

Thus  $\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = a$ .

(c) To get the momentum space wave function, we must do the integral

$$\Phi(p, 0) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \sqrt{\frac{2a^3}{\pi}} \frac{1}{x^2 + a^2} dx \quad (7)$$

Feeding this integral into Maple and telling it to assume  $p$  is real gives an answer that depends on the sign of  $p$ . Simplifying the expression for the two signs gives the single result:

$$\Phi(p, 0) = \sqrt{\frac{a}{\hbar}} e^{-|p|a/\hbar} \quad (8)$$

Note that this is an even function of  $p$ , which is correct, since looking at the integral, the complex exponential is the sum of a real, even function (cosine) and an imaginary, odd function ( $i^*\text{sine}$ ). The product of this exponential with an even function ( $1/(x^2 + a^2)$ ) integrated over a symmetric interval will be non-zero only for the cosine part, and the cosine itself is even in  $p$ , so the result of the integral must also be even in  $p$ .

To check the normalization of  $\Phi(p, 0)$ , we do the integral:

$$\int_{-\infty}^{\infty} |\Phi(p, 0)|^2 dp = 2 \frac{a}{\hbar} \int_0^{\infty} e^{-2pa/\hbar} dp \quad (9)$$

$$= 1 \quad (10)$$

(d) Since  $\Phi(p, 0)$  is even,  $\langle p \rangle = 0$ . For  $\langle p^2 \rangle$ , we have

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} p^2 |\Phi(p, 0)|^2 dp \quad (11)$$

$$= 2 \frac{a}{\hbar} \int_0^{\infty} p^2 e^{-2pa/\hbar} dp \quad (12)$$

$$= \frac{\hbar^2}{2a^2} \quad (13)$$

Thus  $\sigma_p = \hbar/\sqrt{2}a$  and the uncertainty principle is  $\sigma_p \sigma_x = \hbar/\sqrt{2}$ .