MOMENTUM SPACE: ANOTHER EXAMPLE

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Post date: 8 Oct 2012.

Another example of calculations in momentum space. Suppose we have the initial state of a wave function in position space given by

$$\Psi(x,0) = \frac{A}{x^2 + a^2}$$  \hspace{1cm} (1)

where $a$ and $A$ are constants.

(a) Applying normalization

$$1 = A^2 \int_{-\infty}^{\infty} \left( \frac{1}{x^2 + a^2} \right)^2 dx$$  \hspace{1cm} (2)

$$= \frac{\pi A^2}{2a^3}$$  \hspace{1cm} (3)

$$A = \sqrt{\frac{2}{\pi}} a^{3/2}$$  \hspace{1cm} (4)

(b) We can use the position space wave function to work out $\langle x \rangle$ and $\langle x^2 \rangle$.

Since $|\Psi(x,0)|^2 = A^2/(x^2 + a^2)^2$ is even, $\langle x \rangle = 0$. For $\langle x^2 \rangle$, we find

$$\langle x^2 \rangle = \frac{2}{\pi} a^3 \int_{-\infty}^{\infty} x^2 \left( \frac{1}{x^2 + a^2} \right)^2 dx$$  \hspace{1cm} (5)

$$= a^2$$  \hspace{1cm} (6)

Thus $\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = a$.

(c) To get the momentum space wave function, we must do the integral

$$\Phi(p,0) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \sqrt{\frac{2a^3}{\pi}} \frac{1}{x^2 + a^2} dx$$  \hspace{1cm} (7)

Feeding this integral into Maple and telling it to assume $p$ is real gives an answer that depends on the sign of $p$. Simplifying the expression for the two signs gives the single result:
\[ \Phi(p,0) = \sqrt{\frac{\alpha}{\hbar}} e^{-|p|a/\hbar} \]  

(8)

Note that this is an even function of \( p \), which is correct, since looking at the integral, the complex exponential is the sum of a real, even function (cosine) and an imaginary, odd function \( (i \text{sin}) \). The product of this exponential with an even function \( (1/(x^2 + a^2)) \) integrated over a symmetric interval will be non-zero only for the cosine part, and the cosine itself is even in \( p \), so the result of the integral must also be even in \( p \).

To check the normalization of \( \Phi(p,0) \), we do the integral:

\[ \int_{-\infty}^{\infty} |\Phi(p,0)|^2 \, dp = 2 \frac{\alpha}{\hbar} \int_{0}^{\infty} e^{-2pa/\hbar} \, dp \]  

(9)

\[ = 1 \]  

(10)

(d) Since \( \Phi(p,0) \) is even, \( \langle p \rangle = 0 \). For \( \langle p^2 \rangle \), we have

\[ \langle p^2 \rangle = \int_{-\infty}^{\infty} p^2 |\Phi(p,0)|^2 \, dp \]  

(11)

\[ = 2 \frac{\alpha}{\hbar} \int_{0}^{\infty} p^2 e^{-2pa/\hbar} \, dp \]  

(12)

\[ = \frac{\hbar^2}{2a^2} \]  

(13)

Thus \( \sigma_p = \hbar/\sqrt{2\alpha} \) and the uncertainty principle is \( \sigma_p \sigma_x = \hbar/\sqrt{2} \).