HARMONIC OSCILLATOR: MIXTURE OF TWO LOWEST STATES

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 3.34.

Suppose we have a system in the harmonic oscillator potential that starts off as an equal mixture of the lowest two states. The state contains an equal proportion of the ground and first excited states, so we can start with the wave function

$$\Psi(x,t)=\frac{1}{\sqrt{2}}(\psi_0 e^{-iE_0t/\hbar}+\psi_1 e^{-iE_1t/\hbar}) \tag{1}$$

Using the result for the matrix elements of p from the last post, we can calculate $\langle p\rangle$ for this wave function:

$$\begin{aligned}
\langle p\rangle &= \frac{1}{2}(\langle 0|p|1\rangle e^{-i(E_1-E_0)t/\hbar}+\langle 1|p|0\rangle e^{i(E_1-E_0)t/\hbar}) &(2)\\
&= \frac{i}{2}\sqrt{\frac{\hbar m\omega}{2}}(e^{i(E_1-E_0)t/\hbar}-e^{-i(E_1-E_0)t/\hbar}) &(3)\\
&= -\sqrt{\frac{\hbar m\omega}{2}}\sin((E_1-E_0)t/\hbar) &(4)\\
&= -\sqrt{\frac{\hbar m\omega}{2}}\sin\omega t &(5)
\end{aligned}$$

The maximum value of $\langle p\rangle$ is therefore $\sqrt{\frac{\hbar m\omega}{2}}$.

To make this value occur at $t=0$ we need to shift the origin of time by introducing a new time variable τ such that $\tau=t+\pi/2\omega$. Since no observable property of the system can be changed by multiplying each term in 1 by a constant (that is, independent of x) complex exponential, we can merely make this substitution into the original wave function to get

$$\Psi(x,\tau) = \frac{1}{\sqrt{2}}\left(\psi_0 e^{-iE_0(\tau-\pi/2\omega)/\hbar} + \psi_1 e^{-iE_1(\tau-\pi/2\omega)/\hbar}\right) \tag{6}$$

$$= \frac{1}{\sqrt{2}} e^{-i\omega\tau/2}\left[\psi_0 e^{i\pi/4} + \psi_1 e^{-i\omega\tau+3i\pi/4}\right] \tag{7}$$

$$= \frac{1}{2} e^{-i\omega\tau/2}\left((1+i)\,\psi_0 + (-1+i)\,\psi_1 e^{-i\omega\tau}\right) \tag{8}$$

Note that the probability of getting either state is still equal to 0.5 at $\tau = 0$. Making the same substitution in the formula for momentum, we get the expectation value of momentum

$$\langle p\rangle = -\sqrt{\frac{\hbar m\omega}{2}}\sin\omega(\tau-\pi/2\omega) = -\sqrt{\frac{\hbar m\omega}{2}}\sin(\omega\tau-\pi/2) \tag{9}$$

which has its maximum value when $\tau = 0$.