HARMONIC OSCILLATOR: COHERENT STATES

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 3.35.

The uncertainty principle for the nth stationary state of the harmonic oscillator satisfies the condition

$$\sigma_p\sigma_x = \hbar\left(n+\frac{1}{2}\right) \tag{1}$$

Thus only the ground state (where $n=0$) satisfies the uncertainty limit of $\sigma_p\sigma_x = \hbar/2$. However, some linear combinations of the stationary states *do* satisfy this limit. These are called *coherent states* and are eigenstates of the lowering operator a_-. That is, they can be written as

$$a_-|\alpha\rangle = \alpha|\alpha\rangle \tag{2}$$

where α is a complex eigenvalue and the eigenstate $|\alpha\rangle$ is a linear combination of the harmonic oscillator stationary states (well, actually, since these states form a complete set, the last condition is always true, but never mind).

(a) We can calculate the means for position and momentum for these coherent states. We review the equations involving the raising and lowering operators first:

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-) \tag{3}$$

$$p = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-) \tag{4}$$

$$(a_+)^\dagger = a_- \tag{5}$$

We have

$$\langle x\rangle = \langle\alpha|x|\alpha\rangle \tag{6}$$
$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\langle\alpha|a_+|\alpha\rangle+\langle\alpha|a_-|\alpha\rangle\right) \tag{7}$$
$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\langle a_-\alpha|\alpha\rangle+\langle\alpha|a_-|\alpha\rangle\right) \tag{8}$$
$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\alpha^*+\alpha\right) \tag{9}$$
$$= \sqrt{\frac{2\hbar}{m\omega}}\Re(\alpha) \tag{10}$$

where $\Re(\alpha)$ is the real part of α. In getting the third line, we used the hermitian conjugate property of a_+ above.

Doing a similar calculation we can find $\langle x^2\rangle$:

$$\langle x^2\rangle = \frac{\hbar}{2m\omega}\langle\alpha|a_+^2+a_-^2+a_+a_-+a_-a_+|\alpha\rangle \tag{11}$$
$$= \frac{\hbar}{2m\omega}\langle\alpha|a_+^2+a_-^2+a_+a_-+(1+a_+a_-)|\alpha\rangle \tag{12}$$
$$= \frac{\hbar}{2m\omega}(\alpha^{*2}+\alpha^2+2\alpha\alpha^*+1) \tag{13}$$
$$= \frac{\hbar}{m\omega}\left(2\Re(\alpha)^2+\frac{1}{2}\right) \tag{14}$$

where in the second line we have used the commutator $[a_-,a_+]=1$.

The calculations for $\langle p\rangle$ and $\langle p^2\rangle$ are similar and the results are

$$\langle p\rangle = -\sqrt{2\hbar m\omega}\Im(\alpha) \tag{15}$$
$$\langle p^2\rangle = \hbar m\omega\left(2\Im(\alpha)^2+\frac{1}{2}\right) \tag{16}$$

where $\Im(\alpha)$ is the imaginary part of α.

(b) The standard deviations are

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} \tag{17}$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \tag{18}$$

$$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} \tag{19}$$

$$= \sqrt{\frac{\hbar m\omega}{2}} \tag{20}$$

Therefore, the uncertainty principle here is $\sigma_x\sigma_p = \hbar/2$ as required.

(c) The expansion of $|\alpha\rangle$ in terms of the stationary states is $|\alpha\rangle = \sum_{n=0}^{\infty} c_n|n\rangle$, so we have, using the property of the lowering operator $a_-|n\rangle = \sqrt{n}|n-1\rangle$:

$$a_-|\alpha\rangle = \sum_{n=1}^{\infty} c_n\sqrt{n}|n-1\rangle \tag{21}$$

Note that the sum now starts at $n = 1$, since $a_-|0\rangle = 0$. Also, since $a_-|\alpha\rangle = \alpha|\alpha\rangle$:

$$a_-|\alpha\rangle = \sum_{n=0}^{\infty} c_n\alpha|n\rangle \tag{22}$$

$$\sum_{n=1}^{\infty} c_n\sqrt{n}|n-1\rangle = \sum_{n=0}^{\infty} c_n\alpha|n\rangle \tag{23}$$

Since the energy eigenstates form an orthonormal set, we can equate coefficients of each eigenstate. We start with the first term in each series. We have

$$c_1 = \alpha c_0 \tag{24}$$

The next term gives

$$c_2 = \frac{\alpha}{\sqrt{2}}c_1 \tag{25}$$

$$= \frac{\alpha^2}{\sqrt{2}}c_0 \tag{26}$$

The next term is

$$c_3 = \frac{\alpha^3}{\sqrt{3\times 2}}c_0$$

At this point, we can guess the general pattern, which we can propose to be

$$c_n = \frac{\alpha^n}{\sqrt{n!}} c_0 \tag{27}$$

We can prove the formula in the question using mathematical induction. First, we establish the anchor step by equating coefficients of $|0\rangle$:

$$c_1 = \alpha c_0 = \frac{\alpha^1}{\sqrt{1!}} c_0 \tag{28}$$

Now we prove that if $c_n = (\alpha^n/\sqrt{n!})c_0$ then $c_{n+1} = (\alpha^{n+1}/\sqrt{(n+1)!})c_0$. By equating the cofficients of $|n\rangle$ we get

$$\sqrt{n+1}\,c_{n+1} = \alpha c_n \tag{29}$$

$$c_{n+1} = \frac{\alpha}{\sqrt{n+1}} \frac{\alpha^n}{\sqrt{n!}} c_0 \tag{30}$$

$$= \frac{\alpha^{n+1}}{\sqrt{(n+1)!}} c_0 \tag{31}$$

(d) We can find c_0 by normalizing $|\alpha\rangle$. Writing it out in terms of the series:

$$\langle\alpha|\alpha\rangle = \sum_{n=0}^{\infty} c_n^* c_n \langle n|n\rangle \tag{32}$$

$$= \sum_{n=0}^{\infty} |c_n|^2 \tag{33}$$

$$= |c_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} \tag{34}$$

$$= 1 \tag{35}$$

In the first line, we have used the orthogonality of the eigenstates to eliminate all terms containing $\langle n|m\rangle$ where $n \neq m$. In the second line we use the fact that all the energy eigenstates are normalized.

The series is the expansion of $e^{|\alpha|^2}$ so assuming c_0 is real, we get

$$c_0 = e^{-|\alpha|^2/2} \tag{36}$$

(e) Adding in the time dependence, we get

$$a_{-}\sum_{n=0}^{\infty} c_n e^{-iE_n t/\hbar}|n\rangle = \sum_{n=1}^{\infty} c_n \sqrt{n} e^{-iE_n t/\hbar}|n-1\rangle \tag{37}$$

$$= \sum_{n=1}^{\infty} c_n \sqrt{n} e^{-iE_{n-1} t/\hbar} e^{-i\omega t}|n-1\rangle \tag{38}$$

In the second line we have used the fact that for the harmonic oscillator, $E_n = E_{n-1} + \hbar\omega$. This expression can be made equal to $\alpha(t)\sum_{n=0}^{\infty} c_n|n\rangle$ if we define $\alpha(t) \equiv e^{-i\omega t}\alpha$. Thus the time-dependent state is still an eigenstate of a_{-}, but now the eigenvalue is time-dependent.

(f) The harmonic oscillator ground state is a coherent state with eigenvalue $\alpha = 0$, since the lowering operator produces zero when applied to the ground state. We already know that the uncertainty principle is satisfied exactly for the ground state.