

HAMILTONIAN FOR THREE-STATE SYSTEM

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 3.37.

Suppose we have a three-state hamiltonian whose matrix elements, relative to a certain basis, are

$$(1) \quad H = \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & a \end{bmatrix}$$

To find the allowed energies and corresponding eigenstates, we find the eigenvalues and eigenvectors of H .

$$(2) \quad \begin{vmatrix} a-E & 0 & b \\ 0 & c-E & 0 \\ b & 0 & a-E \end{vmatrix} = 0$$

$$(3) \quad (a-E)^2(c-E) - b^2(c-E) = 0$$

This has 3 solutions:

$$(4) \quad E_1 = c$$

$$(5) \quad E_2 = a+b$$

$$(6) \quad E_3 = a-b$$

The corresponding normalized eigenvectors are found in the usual way:

$$(7) \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$(8) \quad |2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$(9) \quad |3\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

If the system starts in the initial state which is the eigenvector $|1\rangle$, the time-dependent solution is

$$(10) \quad |\mathcal{S}(t)\rangle = e^{-iE_1t/\hbar}|1\rangle$$

$$(11) \quad = e^{-ict/\hbar}|1\rangle$$

If the initial state is not one of the eigenvectors, such as

$$(12) \quad |\mathcal{S}(0)\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

we express this state as a linear combination of the last two eigenvectors:

$$(13) \quad |\mathcal{S}(0)\rangle = \frac{1}{\sqrt{2}}(|2\rangle - |3\rangle)$$

so the general solution is

$$(14) \quad |\mathcal{S}(t)\rangle = \frac{1}{\sqrt{2}} \left(e^{-iE_2t/\hbar}|2\rangle - e^{-iE_3t/\hbar}|3\rangle \right)$$

$$(15) \quad = \frac{1}{\sqrt{2}} \left(e^{-i(a+b)t/\hbar}|2\rangle - e^{-i(a-b)t/\hbar}|3\rangle \right)$$

COMMENTS

Remark 1. Chris Kranenberg; Dec 6, 2017 4:46 PM

The solution can be taken a little further by replacing the vector notations with their matrices, perform the matrix addition to obtain a matrix solution with cosine and isine expressions.

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If you mean eqn (15), yes, you could factor out the $e^{-iat/\hbar}$ and then substitute from (8) and (9) to get the answer in terms of sin and cos of bt/\hbar .