

HAMILTONIAN AND OBSERVABLES IN THREE-STATE SYSTEM

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 3.38.

Suppose we have a three-state hamiltonian whose matrix elements, relative to a certain basis, are

$$H = \hbar\omega \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad (1)$$

We also have two other observables A and B , represented by the matrices

$$A = \lambda \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad (2)$$

$$B = \mu \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad (3)$$

Here, ω , λ and μ are positive real numbers.

The hamiltonian is already diagonal, so we can read its eigenvalues from the matrix. We find for H :

$$E = \hbar\omega, 2\hbar\omega \quad (4)$$

where the $2\hbar\omega$ eigenvalue is degenerate (occurs twice). The eigenvector is $[1, 0, 0]$ for $E = \hbar\omega$. For $E = 2\hbar\omega$, the eigenvectors span a two-dimensional space, so we can choose any two linearly independent vectors in that space. The simplest choice is $[0, 1, 0]$ and $[0, 0, 1]$.

The eigenvalues and eigenvectors of A and B are found in the usual way by calculating determinants and solving the resulting polynomial equation.

For A , the eigenvalues are $a = 2\lambda$ and $a = \pm\lambda$. The eigenvectors are $[0, 0, 1]$ for $a = 2\lambda$, $[1, 1, 0]/\sqrt{2}$ for $a = \lambda$ and $[1, -1, 0]/\sqrt{2}$ for $a = -\lambda$.

For B, the eigenvalues are $b = 2\mu$ and $b = \pm\mu$. The eigenvectors are $[1, 0, 0]$ for $b = 2\mu$, $[0, 1, 1]/\sqrt{2}$ for $b = \mu$ and $[0, 1, -1]/\sqrt{2}$ for $b = -\mu$.

Now suppose that the system starts in the state

$$|\mathcal{S}(0)\rangle = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \quad (5)$$

where the state is assumed to be normalized so that $|c_1|^2 + |c_2|^2 + |c_3|^2 = 1$.

We can find the expectation values of the three observables at this time most easily if we can express $|\mathcal{S}(0)\rangle$ as a linear combination of the eigenvectors for each operator. In the case of H, this is easy, and we get

$$|\mathcal{S}(0)\rangle = c_1 H_1 + c_2 H_2 + c_3 H_3 \quad (6)$$

where we have labelled each of the eigenvectors of H as H_1 , etc in the order in which they were calculated above. The expectation value of H is then

$$\langle H \rangle = \langle \mathcal{S}(0) | H | \mathcal{S}(0) \rangle \quad (7)$$

$$= |c_1|^2 \hbar\omega + 2\hbar\omega(|c_2|^2 + |c_3|^2) \quad (8)$$

$$= \hbar\omega(2 - |c_1|^2) \quad (9)$$

For A, we have

$$|\mathcal{S}(0)\rangle = \frac{c_1}{\sqrt{2}}(A_2 + A_3) + \frac{c_2}{\sqrt{2}}(A_2 - A_3) + c_3 A_1 \quad (10)$$

The expectation value is therefore

$$\langle A \rangle = \langle \mathcal{S}(0) | A | \mathcal{S}(0) \rangle \quad (11)$$

$$= 2\lambda|c_3|^2 + \lambda(c_1^* c_2 + c_2^* c_1) \quad (12)$$

$$= 2\lambda(|c_3|^2 + \Re(c_1^* c_2)) \quad (13)$$

For B, we have

$$|\mathcal{S}(0)\rangle = \frac{c_2}{\sqrt{2}}(B_2 + B_3) + \frac{c_3}{\sqrt{2}}(B_2 - B_3) + c_1 B_1 \quad (14)$$

Note that this is a cyclic permutation of the c_i coefficients from 10, so since the B_i are orthonormal, we can read the value for $\langle B \rangle$ from that for $\langle A \rangle$ by cyclically permuting the indices:

$$\langle B \rangle = 2\mu(|c_1|^2 + \Re(c_2^* c_3)) \quad (15)$$

From 6 we can get the time dependent form:

$$|\mathcal{S}(t)\rangle = c_1 e^{-i\omega t} H_1 + e^{-2i\omega t} (c_2 H_2 + c_3 H_3) \quad (16)$$

so a measurement of energy will give $\hbar\omega$ with probability $|c_1|^2$ and $2\hbar\omega$ with probability $|c_2|^2 + |c_3|^2$.

To work out possible values of the other two observables, we note that

$$H_1 = \frac{1}{\sqrt{2}}(A_2 + A_3) \quad (17)$$

$$H_2 = \frac{1}{\sqrt{2}}(A_2 - A_3) \quad (18)$$

$$H_3 = A_1 \quad (19)$$

so from 10, we can write

$$|\mathcal{S}(t)\rangle = \frac{c_1}{\sqrt{2}} e^{-i\omega t} (A_2 + A_3) + \frac{c_2}{\sqrt{2}} e^{-2i\omega t} (A_2 - A_3) + c_3 e^{-2i\omega t} A_1 \quad (20)$$

$$= c_3 e^{-2i\omega t} A_1 + \frac{1}{\sqrt{2}} (c_1 e^{-i\omega t} + c_2 e^{-2i\omega t}) A_2 + \frac{1}{\sqrt{2}} (c_1 e^{-i\omega t} - c_2 e^{-2i\omega t}) A_3 \quad (21)$$

A measurement of A will give 2λ with probability $|c_3|^2$. The second eigenvalue of $+\lambda$ will be obtained with probability

$$p_\lambda = \frac{1}{\sqrt{2}} (c_1^* e^{i\omega t} + c_2^* e^{2i\omega t}) \frac{1}{\sqrt{2}} (c_1 e^{-i\omega t} + c_2 e^{-2i\omega t}) \quad (22)$$

$$= \frac{1}{2} (|c_1|^2 + |c_2|^2 + 2\Re(c_1^* c_2 e^{-i\omega t})) \quad (23)$$

and the probability of the third eigenvalue $-\lambda$ is

$$p_{-\lambda} = \frac{1}{\sqrt{2}} (c_1^* e^{i\omega t} - c_2^* e^{2i\omega t}) \frac{1}{\sqrt{2}} (c_1 e^{-i\omega t} - c_2 e^{-2i\omega t}) \quad (24)$$

$$= \frac{1}{2} (|c_1|^2 + |c_2|^2 - 2\Re(c_1^* c_2 e^{-i\omega t})) \quad (25)$$

Note that the sum of all three probabilities is $|c_1|^2 + |c_2|^2 + |c_3|^2 = 1$. For B , we can reason the same way

$$H_1 = B_1 \quad (26)$$

$$H_2 = \frac{1}{\sqrt{2}}(B_2 + B_3) \quad (27)$$

$$H_3 = \frac{1}{\sqrt{2}}(B_2 - B_3) \quad (28)$$

so

$$\begin{aligned} |\mathcal{S}(t)\rangle &= c_1 e^{-i\omega t} B_1 + \frac{c_2}{\sqrt{2}} e^{-2i\omega t} (B_2 + B_3) + \frac{c_3}{\sqrt{2}} e^{-2i\omega t} (B_2 - B_3) \quad (29) \\ &= c_1 e^{-i\omega t} B_1 + \frac{1}{\sqrt{2}} e^{-2i\omega t} (c_2 + c_3) B_2 + \frac{1}{\sqrt{2}} e^{-2i\omega t} (c_2 - c_3) B_3 \end{aligned} \quad (30)$$

The probability of a measurement of B giving the first eigenvalue of 2μ is therefore $|c_1|^2$. For the second eigenvalue of $+\mu$ the probability is $\frac{1}{2}|c_2 + c_3|^2$ and for the third eigenvalue of $-\mu$ the probability is $\frac{1}{2}|c_2 - c_3|^2$.

Note that $|c_2 + c_3|^2 = |c_2|^2 + |c_3|^2 + 2\Re(c_2^* c_3)$ and $|c_2 - c_3|^2 = |c_2|^2 + |c_3|^2 - 2\Re(c_2^* c_3)$ so again the sum of the three probabilities is $|c_1|^2 + |c_2|^2 + |c_3|^2 = 1$. Note also that only the measurements of A are time dependent.