TRANSLATIONS IN SPACE AND TIME

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 3.39.

Since $\hat{p} = (\hbar/i)\partial/\partial x$, we can write the exponential of the momentum operator by using a Taylor expansion:

$$e^{i\hat{p}x_0/\hbar} = 1 + x_0\frac{\partial}{\partial x} + \frac{x_0^2}{2!}\frac{\partial^2}{\partial x^2} + \ldots \tag{1}$$

where x_0 is a constant. If we apply this operator to a function, we get

$$e^{i\hat{p}x_0/\hbar} f(x) = f(x) + x_0\frac{\partial f}{\partial x} + \frac{x_0^2}{2!}\frac{\partial^2 f}{\partial x^2} + \ldots \tag{2}$$

which is the Taylor expansion of $f(x+x_0)$ about the point x, provided that the derivatives are evaluated at that point. Since this exponential operator effectively shifts the function by a distance x_0, the operator $p/\hbar$ is called the generator of translations in space.

Similarly, for a function $\Psi(x,t)$ that satisfies the time-dependent Schrodinger equation, we have

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t} = \hat{H}\Psi(x,t) \tag{3}$$

so

$$e^{-i\hat{H}t_0/\hbar} = 1 + t_0\frac{\partial}{\partial t} + \frac{t_0^2}{2!}\frac{\partial^2}{\partial t^2} + \ldots \tag{4}$$

and

$$\begin{aligned} e^{-i\hat{H}t_0/\hbar}\Psi(x,t) &= \Psi(x,t) + t_0\frac{\partial\Psi}{\partial t} + \frac{t_0^2}{2!}\frac{\partial^2\Psi}{\partial t^2} + \ldots && (5) \\ &= \Psi(x,t+t_0) && (6) \end{aligned}$$

which is the Taylor expansion with respect to time. The operator $-H/\hbar$ is therefore known as the generator of translations in time.

The mean of an observable at time $t+t_0$ is

$$\begin{aligned}
\langle Q\rangle_{t+t_0} &= \langle\Psi(t+t_0)|Q(t+t_0)|\Psi(t+t_0)\rangle && (7)\\
&= \langle e^{-i\hat{H}t_0/\hbar}\Psi(t)|Q(t+t_0)|e^{-i\hat{H}t_0/\hbar}\Psi(t)\rangle && (8)\\
&= \langle\Psi(t)|e^{i\hat{H}t_0/\hbar}Q(t+t_0)e^{-i\hat{H}t_0/\hbar}|\Psi(t)\rangle && (9)
\end{aligned}$$

Expanding the two exponentials and the operator to first order in t_0, we get

$$\langle Q\rangle_{t+t_0} = \left\langle \Psi(t)|\left(1+\frac{i}{\hbar}t_0\hat{H}\right)\left(Q(t)+t_0\frac{\partial Q}{\partial t}\right)\left(1-\frac{i}{\hbar}t_0\hat{H}\right)|\Psi(t)\right\rangle \quad (10)$$

$$= \left\langle \Psi(t)|Q(t)+t_0\frac{\partial Q}{\partial t}+\frac{i}{\hbar}t_0(\hat{H}Q(t)-Q(t)\hat{H})|\Psi(t)\right\rangle + \mathcal{O}(t_0^2) \quad (11)$$

$$= \langle Q\rangle_t + \left\langle\frac{\partial Q}{\partial t}\right\rangle t_0 + \frac{i}{\hbar}t_0\langle[\hat{H},Q]\rangle + \mathcal{O}(t_0^2) \quad (12)$$

where $\mathcal{O}(t_0^2)$ represents terms of second and higher order in t_0. Moving $\langle Q\rangle_t$ to the LHS, dividing through by t_0 and taking the limit as $t_0\to 0$ gives us the relation we got while studying the energy-time uncertainty principle:

$$\frac{d\langle Q\rangle}{dt} = \left\langle\frac{\partial Q}{\partial t}\right\rangle + \frac{i}{\hbar}\langle[\hat{H},Q]\rangle \quad (13)$$