

TRANSLATIONS IN SPACE AND TIME

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 3.39.

Since $\hat{p} = (\hbar/i)\partial/\partial x$, we can write the exponential of the momentum operator by using a Taylor expansion:

$$(0.1) \quad e^{i\hat{p}x_0/\hbar} = 1 + x_0 \frac{\partial}{\partial x} + \frac{x_0^2}{2!} \frac{\partial^2}{\partial x^2} + \dots$$

where x_0 is a constant. If we apply this operator to a function, we get

$$(0.2) \quad e^{i\hat{p}x_0/\hbar} f(x) = f(x) + x_0 \frac{\partial f}{\partial x} + \frac{x_0^2}{2!} \frac{\partial^2 f}{\partial x^2} + \dots$$

which is the Taylor expansion of $f(x+x_0)$ about the point x , provided that the derivatives are evaluated at that point. Since this exponential operator effectively shifts the function by a distance x_0 , the operator p/\hbar is called the generator of translations in space.

Similarly, for a function $\Psi(x,t)$ that satisfies the time-dependent Schrodinger equation, we have

$$(0.3) \quad i\hbar \frac{\partial \Psi(x,t)}{\partial t} = \hat{H}\Psi(x,t)$$

so

$$(0.4) \quad e^{-i\hat{H}t_0/\hbar} = 1 + t_0 \frac{\partial}{\partial t} + \frac{t_0^2}{2!} \frac{\partial^2}{\partial t^2} + \dots$$

and

$$(0.5) \quad e^{-i\hat{H}t_0/\hbar} \Psi(x,t) = \Psi(x,t) + t_0 \frac{\partial \Psi}{\partial t} + \frac{t_0^2}{2!} \frac{\partial^2 \Psi}{\partial t^2} + \dots$$

$$(0.6) \quad = \Psi(x,t+t_0)$$

which is the Taylor expansion with respect to time. The operator $-H/\hbar$ is therefore known as the generator of translations in time.

The mean of an observable at time $t + t_0$ is

$$(0.7) \quad \langle Q \rangle_{t+t_0} = \langle \Psi(t+t_0) | Q(t+t_0) | \Psi(t+t_0) \rangle$$

$$(0.8) \quad = \langle e^{-i\hat{H}t_0/\hbar} \Psi(t) | Q(t+t_0) | e^{-i\hat{H}t_0/\hbar} \Psi(t) \rangle$$

$$(0.9) \quad = \langle \Psi(t) | e^{i\hat{H}t_0/\hbar} Q(t+t_0) e^{-i\hat{H}t_0/\hbar} | \Psi(t) \rangle$$

Expanding the two exponentials and the operator to first order in t_0 , we get

$$(0.10)$$

$$\langle Q \rangle_{t+t_0} = \left\langle \Psi(t) \left| \left(1 + \frac{i}{\hbar} t_0 \hat{H} \right) \left(Q(t) + t_0 \frac{\partial Q}{\partial t} \right) \left(1 - \frac{i}{\hbar} t_0 \hat{H} \right) \right| \Psi(t) \right\rangle$$

$$(0.11) \quad = \left\langle \Psi(t) \left| Q(t) + t_0 \frac{\partial Q}{\partial t} + \frac{i}{\hbar} t_0 (\hat{H} Q(t) - Q(t) \hat{H}) \right| \Psi(t) \right\rangle + \mathcal{O}(t_0^2)$$

$$(0.12) \quad = \langle Q \rangle_t + \left\langle \frac{\partial Q}{\partial t} \right\rangle t_0 + \frac{i}{\hbar} t_0 \langle [\hat{H}, Q] \rangle + \mathcal{O}(t_0^2)$$

where $\mathcal{O}(t_0^2)$ represents terms of second and higher order in t_0 . Moving $\langle Q \rangle_t$ to the LHS, dividing through by t_0 and taking the limit as $t_0 \rightarrow 0$ gives us the relation we got while studying the energy-time uncertainty principle:

$$(0.13) \quad \frac{d\langle Q \rangle}{dt} = \left\langle \frac{\partial Q}{\partial t} \right\rangle + \frac{i}{\hbar} \langle [\hat{H}, Q] \rangle$$