

FREE PARTICLE IN MOMENTUM SPACE

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 3.40.

Since the Hamiltonian for a free particle is $H = p^2/2m$, the Schrodinger equation in momentum space is

$$i\hbar \frac{\partial \Phi}{\partial t} = \frac{p^2}{2m} \Phi \quad (1)$$

so the solution can be found by simply integrating with respect to t :

$$\Phi(p, t) = e^{-ip^2 t/2m\hbar} \Phi(p, 0) \quad (2)$$

We looked at the travelling Gaussian wave packet in free space earlier. Its initial state in position space is

$$\Psi(x, 0) = \left(\frac{2a}{\pi}\right)^{1/4} e^{-ax^2} e^{ilx} \quad (3)$$

To find $\Phi(p, 0)$ we use the conversion to momentum space we found earlier:

$$\Phi(p, 0) = \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{2a}{\pi}\right)^{1/4} \int_{-\infty}^{\infty} e^{-ax^2 + ilx - ipx/\hbar} dx \quad (4)$$

From the analysis of the travelling Gaussian packet we see that the integral is the same as that done when calculating $\phi(k)$ if we replace k with p/\hbar . Therefore

$$\Phi(p, 0) = \left(\frac{2}{\pi a}\right)^{1/4} \frac{1}{\sqrt{2\hbar}} e^{-(p/\hbar - l)^2/4a} \quad (5)$$

Using 2, we have the full solution for $\Phi(p, t)$:

$$\Phi(p, t) = \left(\frac{2}{\pi a}\right)^{1/4} \frac{1}{\sqrt{2\hbar}} e^{-ip^2 t/2m\hbar} e^{-(p/\hbar - l)^2/4a} \quad (6)$$

Also

$$|\Phi(p, t)|^2 = \frac{1}{\hbar} \frac{1}{\sqrt{2\pi a}} e^{-(p/\hbar - l)^2/2a} \quad (7)$$

which is independent of time. (As a check, we can integrate this over all p and verify that this integral is 1.)

We can calculate the means for momentum in the usual way:

$$\langle p \rangle = \frac{1}{\hbar} \frac{1}{\sqrt{2\pi a}} \int_{-\infty}^{\infty} p e^{-(p/\hbar - l)^2/2a} dp \quad (8)$$

$$= \hbar l \quad (9)$$

$$\langle p^2 \rangle = \frac{1}{\hbar} \frac{1}{\sqrt{2\pi a}} \int_{-\infty}^{\infty} p^2 e^{-(p/\hbar - l)^2/2a} dp \quad (10)$$

$$= \hbar^2 (l^2 + a) \quad (11)$$

Both results agree with those in the analysis of the travelling Gaussian packet.

For the mean energy, we have

$$\langle H \rangle = \left\langle \frac{p^2}{2m} \right\rangle \quad (12)$$

$$= \frac{\hbar^2}{2m} (l^2 + a) \quad (13)$$

$$= \frac{\langle p \rangle^2}{2m} + \frac{a\hbar^2}{2m} \quad (14)$$

Referring back to the stationary Gaussian wave packet in free space, we see that $\langle p^2 \rangle = a\hbar^2$, so the energy is the sum of that for a stationary Gaussian wave packet and the term $\langle p \rangle^2/2m$. For the travelling packet, there is a net non-zero average momentum, so $\langle p \rangle$ is non-zero. Thus the energy arises from the inherent energy of the wave packet, plus the kinetic energy of motion of the packet.