

## UNCERTAINTY PRINCIPLE IN THREE DIMENSIONS

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.1.

In three dimensions, the position and momentum operators are generalizations of their one-dimensional form:

$$(0.1) \quad r_x = x$$

$$(0.2) \quad r_y = y$$

$$(0.3) \quad r_z = z$$

$$(0.4) \quad p_x = -i\hbar \frac{\partial}{\partial x}$$

$$(0.5) \quad p_y = -i\hbar \frac{\partial}{\partial y}$$

$$(0.6) \quad p_z = -i\hbar \frac{\partial}{\partial z}$$

All the position operators commute with each other since they simply multiply. Thus

$$(0.7) \quad [r_i, r_j] = 0$$

Similarly, all the components of momentum commute with each other, since they are derivatives, and there are no occurrences of the variables on which they operate

$$(0.8) \quad [p_i, p_j] = 0$$

Mixtures of momentum and position will interact the same way as in one dimension if the two components are along the same axis. Mixtures of momentum and position that lie along different axes will commute, since the derivative in the momentum is with respect to a component not found in the position. Thus

$$(0.9) \quad [r_i, p_j] = i\hbar \delta_{ij}$$

Earlier, we derived an equation for the rate of change of an observable  $Q$ :

$$(0.10) \quad \frac{d}{dt} \langle Q \rangle = \frac{i}{\hbar} \langle [H, Q] \rangle + \left\langle \frac{\partial Q}{\partial t} \right\rangle$$

where  $H$  is the hamiltonian.

Examining the derivation of this equation shows that nothing depends on the calculation being done in one, two or three dimensions (the wave function and hamiltonian in the derivation could be in any number of dimensions), so it can be applied to each component of  $\mathbf{r}$  and  $\mathbf{p}$  separately. Because of this we can use the results we worked out earlier for the rates of change of position and momentum in one dimension, applied to each of the three axes, and get the corresponding result (Ehrenfest's theorem) in three dimensions:

$$(0.11) \quad \frac{d\langle \mathbf{r} \rangle}{dt} = \frac{\langle \mathbf{p} \rangle}{m}$$

$$(0.12) \quad \frac{d\langle \mathbf{p} \rangle}{dt} = \langle -\nabla V \rangle$$

To work out the uncertainty principle in three dimensions, we can use the relation derived earlier:

$$(0.13) \quad \sigma_A^2 \sigma_B^2 \geq \left( \frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2$$

Again, the derivation of this equation does not depend on the number of dimensions. We can therefore use it to calculate the uncertainty principle for the three components of position and momentum. From the commutators above, we have

$$(0.14) \quad \sigma_{r_i} \sigma_{p_j} \geq \frac{\hbar}{2} \delta_{ij}$$

The  $\delta_{ij}$  indicates that a position component and a perpendicular momentum component can both be measured precisely at the same time, since these components commute, as we saw above. For position and momentum components along the same direction, the uncertainty principle is the same as it is in one dimension.