SPHERICAL HARMONICS - EXAMPLES

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.3.

In the solution of the 3-d Schrödinger equation with a radial potential, we found that the angular part could be written in terms of spherical harmonics

$$Y_l^m(\theta,\phi)=\epsilon\left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2}e^{im\phi}P_l^m(\cos\theta) \tag{1}$$

with $\epsilon=(-1)^m$ for $m>0$ and $\epsilon=1$ for $m<0$ and the P_l^m being the associated Legendre function:

$$P_l^m(x)=(1-x^2)^{m/2}\sum_{k=0}^{[(l-m)/2]}\frac{(2l-2k)!}{2^l(l-k)!k!(l-2k-m)!}(-1)^k x^{l-m-2k} \tag{2}$$

As an example, we can calculate a couple of the spherical harmonics from these formulas.

Y_0^0 can be calculated easily from inspection of the formulas and comes out to

$$Y_0^0=\left(\frac{1}{4\pi}\right)^{1/2} \tag{3}$$

To get Y_2^1, we start with 1:

$$Y_2^1=-\sqrt{\frac{5}{24\pi}}e^{i\phi}P_2^1(\cos\theta) \tag{4}$$

Then from 2

$$\begin{aligned}P_2^1(x)&=(1-x^2)^{1/2}\sum_{k=0}^{[1/2]}\frac{(4-2k)!}{2^2(2-k)!k!(2-2k-1)!}(-1)^k x^{1-2k} &(5)\\ &=(1-x^2)^{1/2}(3x) &(6)\\ P_2^1(\cos\theta)&=3\sin\theta\cos\theta &(7)\end{aligned}$$

So finally we get

$$Y_2^1(\theta,\phi) = -\sqrt{\frac{15}{8\pi}} e^{i\phi} \sin\theta\cos\theta \tag{8}$$

The normalization can be checked by direct integration

$$\int_0^{2\pi}\int_0^{\pi} |Y_0^0|^2 \sin\theta d\theta d\phi = \int_0^{2\pi}\int_0^{\pi} \frac{1}{4\pi}\sin\theta d\theta d\phi \tag{9}$$

$$= 1 \tag{10}$$

$$\int_0^{2\pi}\int_0^{\pi} |Y_2^1|^2 \sin\theta d\theta d\phi = \frac{15}{8\pi}\int_0^{2\pi}\int_0^{\pi} \sin^3\theta\cos^2\theta d\theta d\phi \tag{11}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\phi \tag{12}$$

$$= 1 \tag{13}$$

The orthogonality can be checked in a similar way, by direct integration

$$\int_0^{2\pi}\int_0^{\pi} (Y_0^0)^* Y_2^1 \sin\theta d\theta d\phi = \sqrt{\frac{15}{32\pi^2}}\int_0^{2\pi}\int_0^{\pi} e^{i\phi}\sin^2\theta\cos\theta d\theta d\phi \tag{14}$$

$$= 0 \tag{15}$$

where the zero results from the first integral over θ.

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