ANGULAR EQUATION - ALTERNATIVE SOLUTION

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.4.

In the solution of the 3-d Schrödinger equation with a radial potential, we encountered the differential equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left[l(l+1)-\frac{m^2}{\sin^2\theta}\right]\Theta=0 \tag{1}$$

The general solution of this equation for arbitrary l and m is a spherical harmonic, but for the special case $l=m=0$, there is a second solution:

$$\Theta(\theta)=A\ln\left(\tan\frac{\theta}{2}\right) \tag{2}$$

The equation that must be satisfied when $l=m=0$ is:

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=0 \tag{3}$$

Plugging in the given solution $\Theta(\theta)=A\ln[\tan(\theta/2)]$, we find

$$\begin{aligned}\frac{d\Theta}{d\theta} &= \frac{A}{2}\frac{1}{\tan(\theta/2)}\sec^2(\theta/2) && (4)\\ &= \frac{A}{2}\frac{1}{\sin(\theta/2)\cos(\theta/2)} && (5)\\ &= \frac{A}{\sin\theta} && (6)\end{aligned}$$

Therefore, $\sin\theta(d\Theta/d\theta)=A=$ constant, so its derivative is zero, thus satisfying the differential equation.

The solution is not physically acceptable because it tends to $-\infty$ as $\theta\to 0$ and to $+\infty$ as $\theta\to\pi$, and is therefore not normalizable.