

ANGULAR EQUATION - ALTERNATIVE SOLUTION

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.4.

In the solution of the 3-d Schrödinger equation with a radial potential, we encountered the differential equation

$$(0.1) \quad \frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) + \left[l(l+1) - \frac{m^2}{\sin^2 \theta} \right] \Theta = 0$$

The general solution of this equation for arbitrary l and m is a spherical harmonic, but for the special case $l = m = 0$, there is a second solution:

$$(0.2) \quad \Theta(\theta) = A \ln \left(\tan \frac{\theta}{2} \right)$$

The equation that must be satisfied when $l = m = 0$ is:

$$(0.3) \quad \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) = 0$$

Plugging in the given solution $\Theta(\theta) = A \ln[\tan(\theta/2)]$, we find

$$(0.4) \quad \frac{d\Theta}{d\theta} = \frac{A}{2} \frac{1}{\tan(\theta/2)} \sec^2(\theta/2)$$

$$(0.5) \quad = \frac{A}{2} \frac{1}{\sin(\theta/2) \cos(\theta/2)}$$

$$(0.6) \quad = \frac{A}{\sin \theta}$$

Therefore, $\sin \theta (d\Theta/d\theta) = A = \text{constant}$, so its derivative is zero, thus satisfying the differential equation.

The solution is not physically acceptable because it tends to $-\infty$ as $\theta \rightarrow 0$ and to $+\infty$ as $\theta \rightarrow \pi$, and is therefore not normalizable.