

SPHERICAL HARMONICS - MORE EXAMPLES

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.5.

Another couple of examples of spherical harmonics, which are

$$Y_l^m(\theta, \phi) = \left[\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!} \right]^{1/2} e^{im\phi} P_l^m(\cos\theta) \quad (1)$$

with the P_l^m being the associated Legendre function:

$$P_l^m(x) = (1-x^2)^{m/2} \sum_{k=0}^{[(l-m)/2]} \frac{(2l-2k)!}{2^l(l-k)!k!(l-2k-m)!} (-1)^k x^{l-m-2k} \quad (2)$$

This time, we'll look at Y_l^l and Y_3^2 . To calculate Y_l^l , we start with 1

$$Y_l^l = \sqrt{\frac{2l+1}{4\pi} \frac{1}{(2l)!}} e^{il\phi} P_l^l(\cos\theta) \quad (3)$$

With $l = m$, we get from 2

$$P_l^l(x) = (1-x^2)^{l/2} \sum_{k=0}^0 \frac{(2l-2k)!}{2^l(l-k)!k!(2k)!} (-1)^k x^{-2k} \quad (4)$$

$$= (1-x^2)^{l/2} \frac{(2l)!}{2^l l!} \quad (5)$$

$$P_l^l(\cos\theta) = \frac{\sin^l \theta}{2^l l!} (2l)! \quad (6)$$

Plugging this result into 3 we get

$$Y_l^l(\theta, \phi) = \sqrt{\frac{(2l+1)!}{4\pi} \frac{e^{il\phi}}{2^l l!}} \sin^l \theta \quad (7)$$

The angular differential equation is

$$\frac{1}{\sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{d\Theta}{d\theta} \right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta} \right] \Theta = 0 \quad (8)$$

Multiplying through by $\sin^2 \theta$ we get

$$\sin \theta \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) + [l(l+1) \sin^2 \theta - m^2] \Theta = 0 \quad (9)$$

To test that 7 satisfies the angular equation, we calculate the first term:

$$\sin \theta \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial Y_l^l}{\partial \theta} \right) = \sqrt{\frac{(2l+1)!}{4\pi}} \frac{e^{il\phi}}{2^l l!} l \sin \theta (l \sin^{l-1} \theta \cos^2 \theta - \sin^{l+1} \theta) \quad (10)$$

$$= \sqrt{\frac{(2l+1)!}{4\pi}} \frac{e^{il\phi}}{2^l l!} l (l \sin^l \theta - (l+1) \sin^{l+2} \theta) \quad (11)$$

$$\frac{\partial^2 Y_l^l}{\partial \phi^2} = -\sqrt{\frac{(2l+1)!}{4\pi}} \frac{e^{il\phi}}{2^l l!} l^2 \sin^l \theta \quad (12)$$

$$\sin \theta \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial Y_l^l}{\partial \theta} \right) + \frac{\partial^2 Y_l^l}{\partial \phi^2} = -\sqrt{\frac{(2l+1)!}{4\pi}} \frac{e^{il\phi}}{2^l l!} l(l+1) \sin^{l+2} \theta \quad (13)$$

$$= -l(l+1) \sin^2 \theta Y_l^l \quad (14)$$

Thus the first term cancels off the second term and the equation is satisfied.

For Y_3^2 , we first calculate P_3^2 from 2

$$P_3^2(x) = (1-x^2)^{2/2} \sum_{k=0}^{[1/2]} \frac{(6-2k)!}{2^3 (3-k)! k! (1-2k)!} (-1)^k x^{1-2k} \quad (15)$$

$$= (1-x^2) \frac{6!}{8 \times 3!} x \quad (16)$$

$$P_3^2(\cos \theta) = 15 \sin^2 \theta \cos \theta \quad (17)$$

$$Y_3^2(\theta, \phi) = \sqrt{\frac{105}{32\pi}} e^{2i\phi} \sin^2 \theta \cos \theta \quad (18)$$

By direct calculation we can again verify the angular equation, except this time, since the solution involves ϕ , we need to include the term involving the derivative with respect to ϕ :

$$\sin \theta \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial Y_3^2}{\partial \theta} \right) + l(l+1) \sin^2 \theta Y_3^2 + \frac{\partial^2 Y_3^2}{\partial \phi^2} = 0 \quad (19)$$

We can now work out the two derivative terms:

$$\sin\theta \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial Y_3^2}{\partial\theta} \right) = 4\sqrt{\frac{105}{32\pi}} \sin^2\theta \cos\theta (3\cos^2\theta - 2) e^{2i\phi} \quad (20)$$

$$= \sqrt{\frac{105}{32\pi}} (4\sin^2\theta \cos\theta - 12\sin^4\theta \cos\theta) e^{2i\phi} \quad (21)$$

$$\frac{\partial^2 Y_3^2}{\partial\phi^2} = -4\sqrt{\frac{105}{32\pi}} e^{2i\phi} \sin^2\theta \cos\theta \quad (22)$$

$$\sin\theta \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial Y_3^2}{\partial\theta} \right) + \frac{\partial^2 Y_3^2}{\partial\phi^2} = -12\sqrt{\frac{105}{32\pi}} \sin^4\theta \cos\theta e^{2i\phi} \quad (23)$$

$$= -3(3+1) \sin^2\theta Y_3^2(\theta, \phi) \quad (24)$$

Thus the sum of the two derivative terms cancels the middle term, so Y_3^2 does indeed satisfy the equation.

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