SPHERICAL HARMONICS - MORE EXAMPLES

Link to: [physicspages home page.](#)
To leave a comment or report an error, please use the [auxiliary blog.](#)
Post date: 6 Jan 2013.

Another couple of examples of spherical harmonics, which are

\[
Y_{lm}(\theta, \phi) = \left[ \frac{2l + 1}{2} \frac{(l-m)!}{(l+m)!} \right]^{1/2} \frac{e^{im\phi}}{4\pi} P_{lm}(\cos \theta)
\]

with the \( P_{lm} \) being the associated Legendre function:

\[
P_{lm}(x) = \frac{(1-x^2)^{m/2}}{2^l l!} \sum_{k=0}^{[(l-m)/2]} \frac{(2l-2k)!}{2^l (l-k)! k!(l-2k-m)!} (1-x^2)^{k} (-1)^k x^{l-m-2k}
\]

This time, we’ll look at \( Y_{l}^{l} \) and \( Y_{3}^{2} \). To calculate \( Y_{l}^{l} \), we start with

\[
Y_{l}^{l}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} e^{il\phi} P_{l}(\cos \theta)
\]

With \( l = m \), we get from

\[
P_{l}(x) = (1-x^2)^{\frac{m}{2}} \sum_{k=0}^{0} \frac{(2l-2k)!}{2^l (l-k)! k!(2k)!} (-1)^k x^{2k}
\]

\[
= (1-x^2)^{\frac{m}{2}} \frac{2^l}{2^l m!} x^{-2}\]

\[
P_{l}(\cos \theta) = \frac{\sin^l \theta}{2^l l!} (2l)!
\]

Plugging this result into we get

\[
Y_{l}^{l}(\theta, \phi) = \sqrt{\frac{(2l+1)!}{4\pi}} e^{il\phi} \sin^l \theta
\]

The [angular differential equation](#) is

\[
\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) + \left[ l(l+1) - \frac{m^2}{\sin^2 \theta} \right] \Theta = 0
\]

(8)
Multiplying through by $\sin^2 \theta$ we get

$$\sin \theta \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) + [l(l+1) \sin^2 \theta - m^2] \Theta = 0 \quad (9)$$

To test that $\Theta$ satisfies the angular equation, we calculate the first term:

$$\sin \theta \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial Y^l_m}{\partial \theta} \right) = \sqrt{\frac{(2l+1)!}{4\pi}} \frac{e^{il\phi}}{2^l l!} l \sin \left( l \cos^2 \theta - \sin \theta \right) \quad (10)$$

$$= \sqrt{\frac{(2l+1)!}{4\pi}} \frac{e^{il\phi}}{2^l l!} l \sin \left( l \sin \theta \right) \quad (11)$$

$$\frac{\partial^2 Y^l_m}{\partial \phi^2} = -\frac{(2l+1)!}{4\pi} \frac{e^{il\phi}}{2^l l!} l^2 \sin \left( l \sin \theta \right) \quad (12)$$

$$\sin \theta \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial Y^l_m}{\partial \theta} \right) + \frac{\partial^2 Y^l_m}{\partial \phi^2} = -l(l+1) \sin \theta Y^l_m \quad (13)$$

Thus the first term cancels off the second term and the equation is satisfied.

For $Y^2_3$, we first calculate $P^2_3$ from $\Theta$.

$$P^2_3(x) = (1-x^2)^{2/2} \sum_{k=0}^{[1/2]} (6 - 2k)! \frac{2^k (3-k)! k! (1-2k)!}{2^{3k} k!} (-1)^k x^{1-2k} \quad (15)$$

$$= (1-x^2)^2 \frac{6!}{8 \times 3!} x \quad (16)$$

$$P^2_3(\cos \theta) = 15 \sin^2 \theta \cos \theta \quad (17)$$

$$Y^2_3(\theta, \phi) = \sqrt{\frac{105}{32\pi}} e^{2i\phi} \sin^2 \theta \cos \theta \quad (18)$$

By direct calculation we can again verify the angular equation, except this time, since the solution involves $\phi$, we need to include the term involving the derivative with respect to $\phi$:

$$\sin \theta \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial Y^2_3}{\partial \theta} \right) + l(l+1) \sin^2 \theta Y^2_3 + \frac{\partial^2 Y^2_3}{\partial \phi^2} = 0 \quad (19)$$

We can now work out the two derivative terms:
SPHERICAL HARMONICS - MORE EXAMPLES

\[
\sin \theta \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial Y^2_3}{\partial \theta} \right) = 4 \sqrt{\frac{105}{32\pi}} \sin^2 \theta \cos \theta (3 \cos^2 \theta - 2)e^{2i\phi} 
\]

(20)

\[
= \sqrt{\frac{105}{32\pi}} (4 \sin^2 \theta \cos \theta - 12 \sin^4 \theta \cos \theta) e^{2i\phi} 
\]

(21)

\[
\frac{\partial^2 Y^2_3}{\partial \phi^2} = -\frac{4}{\sqrt{105}} e^{2i\phi} \sin^2 \theta \cos \theta 
\]

(22)

\[
\sin \theta \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial Y^2_3}{\partial \theta} \right) + \frac{\partial^2 Y^2_3}{\partial \phi^2} = -12 \sqrt{\frac{105}{32\pi}} \sin^4 \theta \cos \theta e^{2i\phi} 
\]

(23)

\[
= -3(3 + 1) \sin^2 \theta Y^2_3(\theta, \phi) 
\]

(24)

Thus the sum of the two derivative terms cancels the middle term, so \(Y^2_3\) does indeed satisfy the equation.

PINGBACKS

Pingback: **Spherical harmonic at the top of the ladder**
Pingback: **Spherical harmonics: normalization**