SPHERICAL HARMONICS - MORE EXAMPLES

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.5.

Another couple of examples of spherical harmonics, which are

$$Y_l^m(\theta,\phi)=\left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2}e^{im\phi}P_l^m(\cos\theta) \tag{0.1}$$

with the P_l^m being the associated Legendre function:

$$P_l^m(x)=(1-x^2)^{m/2}\sum_{k=0}^{[(l-m)/2]}\frac{(2l-2k)!}{2^l(l-k)!k!(l-2k-m)!}(-1)^k x^{l-m-2k} \tag{0.2}$$

This time, we'll look at Y_l^l and Y_3^2. To calculate Y_l^l, we start with 0.1

$$Y_l^l=\sqrt{\frac{2l+1}{4\pi}\frac{1}{(2l)!}}e^{il\phi}P_l^l(\cos\theta) \tag{0.3}$$

With $l=m$, we get from 0.2

$$\begin{aligned}
P_l^l(x) &= (1-x^2)^{l/2}\sum_{k=0}^{0}\frac{(2l-2k)!}{2^l(l-k)!k!(2k)!}(-1)^k x^{-2k} && (0.4)\\
&= (1-x^2)^{l/2}\frac{(2l)!}{2^l l!} && (0.5)\\
P_l^l(\cos\theta) &= \frac{\sin^l\theta}{2^l l!}(2l)! && (0.6)
\end{aligned}$$

Plugging this result into 0.3 we get

$$Y_l^l(\theta,\phi)=\sqrt{\frac{(2l+1)!}{4\pi}}\frac{e^{il\phi}}{2^l l!}\sin^l\theta \tag{0.7}$$

The angular differential equation is

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left[l(l+1)-\frac{m^2}{\sin^2\theta}\right]\Theta=0 \tag{0.8}$$

Multiplying through by $\sin^2\theta$ we get

$$\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left[l(l+1)\sin^2\theta-m^2\right]\Theta=0 \tag{0.9}$$

To test that 0.7 satisfies the angular equation, we calculate the first term:

$$\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_l^l}{\partial\theta}\right)=\sqrt{\frac{(2l+1)!}{4\pi}}\frac{e^{il\phi}}{2^l l!}l\sin\theta(l\sin^{l-1}\theta\cos^2\theta-\sin^{l+1}\theta) \tag{0.10}$$

$$=\sqrt{\frac{(2l+1)!}{4\pi}}\frac{e^{il\phi}}{2^l l!}l(l\sin^l\theta-(l+1)\sin^{l+2}\theta) \tag{0.11}$$

$$\frac{\partial^2 Y_l^l}{\partial\phi^2}=-\sqrt{\frac{(2l+1)!}{4\pi}}\frac{e^{il\phi}}{2^l l!}l^2\sin^l\theta \tag{0.12}$$

$$\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_l^l}{\partial\theta}\right)+\frac{\partial^2 Y_l^l}{\partial\phi^2}=-\sqrt{\frac{(2l+1)!}{4\pi}}\frac{e^{il\phi}}{2^l l!}l(l+1)\sin^{l+2}\theta \tag{0.13}$$

$$=-l(l+1)\sin^2\theta Y_l^l \tag{0.14}$$

Thus the first term cancels off the second term and the equation is satisfied.

For Y_3^2, we first calculate P_3^2 from 0.2

$$P_3^2(x)=(1-x^2)^{2/2}\sum_{k=0}^{[1/2]}\frac{(6-2k)!}{2^3(3-k)!k!(1-2k)!}(-1)^k x^{1-2k} \tag{0.15}$$

$$=(1-x^2)\frac{6!}{8\times 3!}x \tag{0.16}$$

$$P_3^2(\cos\theta)=15\sin^2\theta\cos\theta \tag{0.17}$$

$$Y_3^2(\theta,\phi)=\sqrt{\frac{105}{32\pi}}e^{2i\phi}\sin^2\theta\cos\theta \tag{0.18}$$

By direct calculation we can again verify the angular equation, except this time, since the solution involves ϕ, we need to include the term involving the derivative with respect to ϕ:

$$\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_3^2}{\partial\theta}\right)+l(l+1)\sin^2\theta Y_3^2+\frac{\partial^2 Y_3^2}{\partial\phi^2}=0 \tag{0.19}$$

We can now work out the two derivative terms:

$$\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_3^2}{\partial\theta}\right)=4\sqrt{\frac{105}{32\pi}}\sin^2\theta\cos\theta(3\cos^2\theta-2)e^{2i\phi} \tag{0.20}$$

$$=\sqrt{\frac{105}{32\pi}}(4\sin^2\theta\cos\theta-12\sin^4\theta\cos\theta)e^{2i\phi} \tag{0.21}$$

$$\frac{\partial^2 Y_3^2}{\partial\phi^2}=-4\sqrt{\frac{105}{32\pi}}e^{2i\phi}\sin^2\theta\cos\theta \tag{0.22}$$

$$\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_3^2}{\partial\theta}\right)+\frac{\partial^2 Y_3^2}{\partial\phi^2}=-12\sqrt{\frac{105}{32\pi}}\sin^4\theta\cos\theta e^{2i\phi} \tag{0.23}$$

$$=-3(3+1)\sin^2\theta Y_3^2(\theta,\phi) \tag{0.24}$$

Thus the sum of the two derivative terms cancels the middle term, so Y_3^2 does indeed satisfy the equation.

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