

HYDROGEN ATOM - LAGUERRE POLYNOMIALS EXAMPLE

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.12.

The radial function for the hydrogen atom wave function can be written in terms of associated Laguerre polynomials L_n^k .

$$(0.1) \quad R_{nl}(r) = \frac{1}{r} \rho^{l+1} e^{-\rho} L_{n-l-1}^{2l+1}(2\rho)$$

Although there is a formula giving the associated Laguerre polynomials in terms of ordinary Laguerre polynomials, which are in turn given in terms of derivatives, it is easier to work out the associated Laguerre polynomials using the explicit series formula:

$$(0.2) \quad L_p^q(x) = c_0 \sum_{j=0}^p \frac{(-1)^j (p+q)!}{(p-j)!(q+j)!j!} x^j$$

This formula gives the polynomial up to an overall normalization constant c_0 .

For example, when $n = 5$ and $l = 2$, we get

$$(0.3) \quad L_{n-l-1}^{2l+1}(2\rho) = L_2^5(2\rho)$$

$$(0.4) \quad = 7!c_0 \sum_{j=0}^2 \frac{(-2)^j}{(2-j)!(5+j)!j!} \rho^j$$

$$(0.5) \quad = 7!c_0 \left(\frac{1}{2!5!} - \frac{2}{6!}\rho + \frac{4}{7!2!}\rho^2 \right)$$

$$(0.6) \quad = c_0 (21 - 14\rho + 2\rho^2)$$

We can also use the recursion formula for $n = 5$ and $l = 2$ which is

$$(0.7) \quad c_{j+1} = \frac{2(j-2)}{(j+1)(j+6)} c_j$$

We get

$$(0.8) \quad c_1 = -\frac{2}{3}c_0$$

$$(0.9) \quad c_2 = -\frac{1}{7}c_1 = \frac{2}{21}c_0$$

This agrees with the values above.

The radial function in this case is (since $\rho = r/5a$)

$$(0.10) \quad R_{52}(r) = \frac{r^2}{(5a)^3} e^{-r/5a} \left(1 - \frac{2}{15a}r + \frac{2}{525a^2}r^2 \right) c_0$$

Normalizing gives

$$(0.11) \quad c_0 = \sqrt{\frac{56}{125a}}$$