EARTH-SUN SYSTEM AS A QUANTUM ATOM

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.17.

An amusing little problem is to consider the Earth-Sun system as a giant analogue to the quantum model of the hydrogen atom. Since the gravitational potential is inverse-r, the same as the electromagnetic force, the calculations translate quite easily. The potential is

$$V=-G\frac{mM}{r} \tag{1}$$

where G is the gravitational constant, $G=6.673\times10^{-11}\,m^{3}kg^{-1}s^{-2}$.

The Bohr radius for the Earth can be found by replacing e^{2} by mM, and $1/4\pi\varepsilon_{0}$ by G.

$$a_{g}=\frac{\hbar^{2}}{Gm^{2}M} \tag{2}$$

In MKS, the values are $\hbar=1.0546\times10^{-34}\,m^{2}kg/s$, mass of the Earth is $m=5.9742\times10^{24}$ kg, and mass of the Sun is $M=1.98892\times10^{30}$ kg. Plugging the numbers gives

$$a_{g}=2.349\times10^{-138}\,m \tag{3}$$

(that is, small).

From Bohr's formula for the energy levels, doing the same replacements as in above, we get

$$E_{n}=-\left[\frac{m}{2\hbar^{2}}(GmM)^{2}\right]\frac{1}{n^{2}} \tag{4}$$

In classical physics, the energy of a planet in a circular orbit of radius r_{0} is the sum of the gravitational potential energy and the kinetic energy. The gravitational potential is

$$V_g = -\frac{GmM}{r_0} \tag{5}$$

The kinetic energy can be obtained from equating the planet's centripetal force mv^2/r_0 to the gravitational force GmM/r_0^2.

$$\frac{mv^2}{r_0} = \frac{GmM}{r_0^2} \tag{6}$$

$$\frac{1}{2}mv^2 = \frac{GmM}{2r_0} \tag{7}$$

$$E = V_g + \frac{1}{2}mv^2 \tag{8}$$

$$= -\frac{GmM}{2r_0} \tag{9}$$

Equating this classical version of the energy with the Bohr energy from 4 we get

$$n^2 = \frac{Gm^2M}{\hbar^2}r_0 \tag{10}$$

$$= \frac{r_0}{a_g} \tag{11}$$

using 2. For the Earth, $r_0 = 1.49598 \times 10^{11}$ m, so the Earth's quantum number is around 2.524×10^{74}.

The difference between energy levels for the Earth is very small, so we can use an approximation:

$$\frac{1}{(n-1)^2} - \frac{1}{n^2} = \frac{n^2-(n-1)^2}{n^2(n-1)^2} \tag{12}$$

$$= \frac{2n-1}{n^2(n-1)^2} \tag{13}$$

$$= \frac{2-\frac{1}{n}}{n(n-1)^2} \tag{14}$$

$$\approx \frac{2}{n^3} \tag{15}$$

for large n. Therefore, the energy released by a transition to the next lower level is

$$\Delta E = \frac{m(GmM)^2}{2\hbar^2}\frac{2}{n^3} \tag{16}$$

$$= 2.1 \times 10^{-41} J \tag{17}$$

The frequency of the photon is thus $\nu = \Delta E/h = 3.168 \times 10^{-8} sec^{-1}$, and the wavelength is

$$\lambda = \frac{c}{\nu} \tag{18}$$

$$= 9.469 \times 10^{15} m \tag{19}$$

$$\approx 1\, l.y. \tag{20}$$

This value results from the fact that the Earth's frequency of oscillation is once per year, as that is the time it takes for the Earth to complete a single orbit. The photon or graviton takes a full year to be emitted, so its wavelength will be 1 light year.