EARTH-SUN SYSTEM AS A QUANTUM ATOM

An amusing little problem is to consider the Earth-Sun system as a giant analogue to the quantum model of the hydrogen atom. Since the gravitational potential is inverse-\(r\), the same as the electromagnetic force, the calculations translate quite easily. The potential is

\[
V = -G \frac{mM}{r}
\]  

(1)

where \(G\) is the gravitational constant, \(G = 6.673 \times 10^{-11} m^3 kg^{-1} s^{-2}\).

The Bohr radius for the Earth can be found by replacing \(e^2\) by \(mM\), and \(1/4\pi\varepsilon_0\) by \(G\).

\[
a_g = \frac{\hbar^2}{Gm^2M}
\]  

(2)

In MKS, the values are \(\hbar = 1.0546 \times 10^{-34} m^2 kg/s\), mass of the Earth is \(m = 5.9742 \times 10^{24}\) kg, and mass of the Sun is \(M = 1.98892 \times 10^{30}\) kg. Plugging the numbers gives

\[
a_g = 2.349 \times 10^{-138} m
\]  

(3)

(that is, small).

From Bohr’s formula for the energy levels, doing the same replacements as in above, we get

\[
E_n = - \left[ \frac{m}{2\hbar^2} (GmM)^2 \right] \frac{1}{n^2}
\]  

(4)

In classical physics, the energy of a planet in a circular orbit of radius \(r_0\) is the sum of the gravitational potential energy and the kinetic energy. The gravitational potential is

\[
V_g = - \frac{GmM}{r_0}
\]  

(5)
The kinetic energy can be obtained from equating the planet’s centripetal force \( \frac{mv^2}{r_0} \) to the gravitational force \( \frac{GmM}{r_0^2} \).

\[
\frac{mv^2}{r_0} = \frac{GmM}{r_0^2} \quad (6)
\]

\[
\frac{1}{2}mv^2 = \frac{GmM}{2r_0} \quad (7)
\]

\[
E = V_0 + \frac{1}{2}mv^2 = -\frac{GmM}{2r_0} \quad (8)
\]

Equating this classical version of the energy with the Bohr energy from 4 we get

\[
n^2 = \frac{Gm^2M}{\hbar^2r_0} = \frac{r_0}{a_0} \quad (10)
\]

\[
\quad \quad \quad = \frac{r_0}{a_g} \quad (11)
\]

using 2. For the Earth, \( r_0 = 1.49598 \times 10^{11} \) m, so the Earth’s quantum number is around \( 2.524 \times 10^{74} \).

The difference between energy levels for the Earth is very small, so we can use an approximation:

\[
\frac{1}{(n-1)^2} - \frac{1}{n^2} = \frac{n^2 - (n-1)^2}{n^2(n-1)^2} = \frac{2n - 1}{n^2(n-1)^2} \approx \frac{2}{n^3} \quad (12)
\]

\[
\approx \frac{2}{n^3} \quad (13)
\]

\[
\approx \frac{2}{n^3} \quad (14)
\]

\[
\approx \frac{2}{n^3} \quad (15)
\]

for large \( n \). Therefore, the energy released by a transition to the next lower level is

\[
\Delta E = \frac{m(GmM)^2}{2\hbar^2} \frac{2}{n^3} \quad (16)
\]

\[
= 2.1 \times 10^{-41} \text{ J} \quad (17)
\]
The frequency of the photon is thus $\nu = \Delta E/h = 3.168 \times 10^{-8} \text{sec}^{-1}$, and the wavelength is

$$\lambda = \frac{c}{\nu}$$

$$= 9.469 \times 10^{15} \text{m}$$

$$\approx 1 \text{ l.y.}$$

This value results from the fact that the Earth’s frequency of oscillation is once per year, as that is the time it takes for the Earth to complete a single orbit. The photon or graviton takes a full year to be emitted, so its wavelength will be 1 light year.