ANGULAR MOMENTUM - RAISING AND LOWERING OPERATORS

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education Section 4.3.1 & Problem 4.18.

The angular momentum operator **L** in quantum mechanics has three components that are not mutually observable. In the calculation of the eigenvalues of L^2 and L_z, we made use of the raising and lowering operators $L_{\pm}$, defined as follows:

$$L_{\pm} \equiv L_x \pm iL_y \tag{1}$$

We showed that the effect of these operators on an eigenfunction f_l^m of L^2 and L_z is to generate a new eigenfunction with the properties

$$\begin{aligned} L^2(L_{\pm}f_l^m) &= \hbar^2 l(l+1)(L_{\pm}f_l^m) \quad (2)\\ L_z(L_{\pm}f_l^m) &= \hbar(m\pm1)(L_{\pm}f_l^m) \quad (3)\end{aligned}$$

That is, the new eigenfunction has the same eigenvalue as the original eigenfunction with respect to the operator L^2 and an eigenvalue raised or lowered by $\hbar$ with respect to L_z.

It is useful to derive a formula which gives the exact operation of the raising and lowering operators on the eigenfunctions. That is, we seek a formula of form

$$L_{\pm}f_l^m = A_l^m f_l^{m\pm1} \tag{4}$$

for some constant A_l^m. It turns out we can find the value of this constant without knowing anything more about the eigenfunctions than that they are normalized.

To do this, we need to show that $L_{\mp}$ is the hermitian conjugate of $L_{\pm}$. To be strict about this, we need first to show that L_x and L_y are hermitian. Since they are observables, they have to be, but we can show it explicitly anyway.

To prove that the components of angular momentum are Hermitian, we can use the same technique as that used for proving that linear momentum is Hermitian. The difference here is that matrix elements are integrated over

3 dimensions rather than one. We can see how the proof goes for the case of $L_x = yp_z - zp_y = -i\hbar y\partial/\partial z + i\hbar z\partial/\partial y$.

$$\langle g|L_x f\rangle = -i\hbar\int\int\int g^*\left(y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y}\right)dxdydz \tag{5}$$

We can consider the first term in this integral, since all other terms in this and the other integrals for the other components work in the same way. Using integration by parts over z we get

$$\int\int\int g^* y\frac{\partial f}{\partial z}dzdydz = \int\int g^* y(f|_{z=-\infty}^{z=\infty})dydx - \int\int\int y\frac{\partial g^*}{\partial z}fdzdydx \tag{6}$$

The first term is zero for the usual reason that any function f that is physically meaningful must be normalizable and so must go to zero at infinity. The second term uses the fact that y is a constant when taking the derivative with respect to z. If we plug this result and the similar one for the second term in the integral back into the original equation, we get

$$\begin{aligned}\langle g|L_x f\rangle &= -i\hbar\int\int\int g^*\left(y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y}\right)dxdydz & (7)\\ &= i\hbar\int\int\int f\left(y\frac{\partial g^*}{\partial z} - z\frac{\partial g^*}{\partial y}\right)dxdydz & (8)\\ &= \langle L_x g|f\rangle & (9)\end{aligned}$$

Once we know that L_x and L_y are Hermitian, then it follows that $L_\mp^\dagger = (L_x \mp i\hbar L_y)^\dagger = L_x \pm i\hbar L_y = L_\pm$.

Now to get the constant A_l^m, we have

$$\begin{aligned}\langle L_\pm f_l^m|L_\pm f_l^m\rangle &= \langle f_l^m|L_\mp L_\pm f_l^m\rangle & (10)\\ &= \langle f_l^m|(L^2 - L_z^2 \mp \hbar L_z)f_l^m\rangle & (11)\\ &= \hbar^2(l(l+1) - m^2 \mp m)\langle f_l^m|f_l^m\rangle & (12)\\ &= (A_l^m)^2\langle f_l^{m\pm1}|f_l^{m\pm1}\rangle & (13)\end{aligned}$$

The second line uses the equation derived earlier:

$$L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z \tag{14}$$

and the third line uses the eigenvalues of L^2 and L_z.

Assuming all the f_l^m are normalized,

$$A_l^m = \hbar\sqrt{l(l+1)-m^2\mp m} \tag{15}$$
$$= \hbar\sqrt{(l\mp m)(l\pm m+1)} \tag{16}$$

Applying L_+ to f_l^l or L_- to f_l^{-l} results in A_l^m being zero, as required.

COMMENTS

Nick; Dec 6, 2017 1:51 AM

I can't follow from eq. 12 to 13. What justifies adding or subtracting 1 from m on the f state?

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The LHS of (10) is equal to (13) because of (4), so (13) is equal to (12).

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