

## ANGULAR MOMENTUM - COMMUTATORS WITH POSITION AND MOMENTUM

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education Problem 4.19.

We've worked out the commutators of the components of angular momentum with each other, but it's also instructive to see what the commutators of angular momentum with position and linear momentum are.

The commutators are all derived similarly, so here are a couple of them:

$$\begin{aligned}
 (0.1) \quad [L_z, x] &= [xp_y - yp_x, x] \\
 (0.2) \quad &= -y[p_x, x] \\
 (0.3) \quad &= i\hbar y \\
 (0.4) \quad [L_z, p_x] &= [xp_y - yp_x, p_x] \\
 (0.5) \quad &= [xp_y, p_x] \\
 (0.6) \quad &= [x, p_x]p_y \\
 (0.7) \quad &= i\hbar p_y
 \end{aligned}$$

We've used the position-momentum commutator:  $[x, p_x] = i\hbar$ , with similar expressions for  $y$  and  $z$ . The complete set of results is

$$\begin{aligned}
 (0.8) \quad [L_z, x] &= i\hbar y \\
 (0.9) \quad [L_z, y] &= -i\hbar x \\
 (0.10) \quad [L_z, z] &= 0 \\
 (0.11) \quad [L_z, p_x] &= i\hbar p_y \\
 (0.12) \quad [L_z, p_y] &= -i\hbar p_x \\
 (0.13) \quad [L_z, p_z] &= 0
 \end{aligned}$$

We can use these results to derive the original commutator:

$$\begin{aligned}
 (0.14) \quad [L_z, L_x] &= [L_z, yp_z - zp_y] \\
 (0.15) \quad &= [L_z, y]p_z - z[L_z, p_y] \\
 (0.16) \quad &= -i\hbar xp_z + i\hbar zp_x \\
 (0.17) \quad &= i\hbar L_y
 \end{aligned}$$

We can now find the commutator of  $L_z$  with the square of the position  $r^2$ . To find the commutator, we apply it to some function  $f$ . Subscripts on  $f$  indicate derivatives w.r.t. that variable; thus  $\partial f/\partial x \equiv f_x$ , etc.

$$(0.18) \quad [L_z, r^2] f = [xp_y - yp_x, r^2] f$$

$$(0.19) \quad = -i\hbar (2xyf + xr^2f_y - 2xyf - yr^2f_x - r^2xf_y + r^2yf_x)$$

$$(0.20) \quad = 0$$

For  $p^2$  we have

$$(0.21) \quad [L_z, p^2] = [xp_y - yp_x, p^2] f$$

$$(0.22) \quad = [xp_y - yp_x, p_x^2] f + [xp_y - yp_x, p_y^2] f$$

$$(0.23) \quad = [xp_y, p_x^2] f - [yp_x, p_y^2] f$$

In the second line we have eliminated  $p_z^2$  since it commutes with  $L_z$  as  $L_z$  contains no reference to  $z$ . Similarly, to get the third line, we have eliminated those terms in the second line that have a zero commutator.

To evaluate the last line, we note that

$$(0.24) \quad [xp_y, p_x^2] f = -\frac{\hbar^3}{i} \left( xf_{yxx} - \frac{\partial^2}{\partial x^2}(xf_y) \right)$$

$$(0.25) \quad = -\frac{\hbar^3}{i} (xf_{yxx} - f_{yx} - f_{xy} - xf_{yxx})$$

$$(0.26) \quad = -\frac{\hbar^3}{i} (-f_{yx} - f_{xy})$$

$$(0.27) \quad [yp_x, p_y^2] f = -\frac{\hbar^3}{i} (yf_{xyy} - f_{yx} - f_{xy} - yf_{xyy})$$

$$(0.28) \quad = -\frac{\hbar^3}{i} (-f_{yx} - f_{xy})$$

Combining all the terms we get

$$(0.29) \quad [L_z, p^2] = 0$$

By symmetry, the same argument shows that  $L_x$  and  $L_y$  also commute with  $r^2$  and  $p^2$  so it follows that all components of  $\mathbf{L}$  commute with  $H = p^2/2m + V$ , if  $V$  depends only on  $r$ .

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