ANGULAR MOMENTUM AND TORQUE

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education Problem 4.20.

In classical physics, the rotational analogue of Newton's law is

$$\frac{d\mathbf{L}}{dt}=\mathbf{N} \tag{1}$$

where $\mathbf{N}$ is the applied torque, and is given in terms of the force by

$$\mathbf{N}=\mathbf{r}\times\mathbf{F} \tag{2}$$

In quantum mechanics, we can apply the relation we worked out earlier for the rate of change of an operator, with $\hat{Q}=\mathbf{L}$. We will show the derivation for the x component, since the other two are similar. The rate of change equation is

$$\frac{d}{dt}\langle Q\rangle=\frac{i}{\hbar}\langle[H,Q]\rangle+\left\langle\frac{\partial Q}{\partial t}\right\rangle \tag{3}$$

We know from the previous post that all three components of $\mathbf{L}$ commute with p^2, so the commutator is

$$\frac{i}{\hbar}[H,L_x] = \frac{i}{\hbar}[V,yp_z-zp_y] \tag{4}$$

Although we also saw in the last post that $\mathbf{L}$ commutes with V if the potential depends only on r, this doesn't help us here, since in order for the torque to be non-zero, the potential would have to have a non-radial dependence, since if $V=V(r)$, then $\mathbf{F}=-\nabla V=-\frac{dV}{dr}\hat{\mathbf{r}}$ and $\mathbf{N}=\mathbf{r}\times\left(-\frac{dV}{dr}\hat{\mathbf{r}}\right)=0$.

As usual, we introduce an auxiliary function f to give the operators something to operate on:

$$\frac{i}{\hbar}[H,L_x]f = \frac{i}{\hbar}[V,yp_z - zp_y]f \tag{5}$$

$$= V\left(y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y}\right) + z\frac{\partial (fV)}{\partial y} - y\frac{\partial (fV)}{\partial z} \tag{6}$$

$$= \left(z\frac{\partial V}{\partial y} - y\frac{\partial V}{\partial z}\right) f \tag{7}$$

Writing out the components of the torque $\mathbf{N} = \mathbf{r} \times (-\nabla V)$, we find that the last line above is equivalent to

$$\frac{i}{\hbar}[H,L_x]f = N_x f \tag{8}$$

Assuming there is no explicit time dependence in the system, this gives the result

$$\left\langle \frac{d\mathbf{L}}{dt} \right\rangle = \langle \mathbf{r} \times \mathbf{F} \rangle \tag{9}$$

$$= \langle \mathbf{N} \rangle \tag{10}$$