

ANGULAR MOMENTUM AND TORQUE

Link to: [physicspages home page](#).

To leave a comment or report an error, please use the [auxiliary blog](#).

References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education Problem 4.20.

In classical physics, the rotational analogue of Newton's law is

$$(0.1) \quad \frac{d\mathbf{L}}{dt} = \mathbf{N}$$

where \mathbf{N} is the applied torque, and is given in terms of the force by

$$(0.2) \quad \mathbf{N} = \mathbf{r} \times \mathbf{F}$$

In quantum mechanics, we can apply the relation we worked out earlier for the rate of change of an operator, with $\hat{Q} = \mathbf{L}$. We will show the derivation for the x component, since the other two are similar. The rate of change equation is

$$(0.3) \quad \frac{d}{dt} \langle Q \rangle = \frac{i}{\hbar} \langle [H, Q] \rangle + \left\langle \frac{\partial Q}{\partial t} \right\rangle$$

We know from the previous post that all three components of \mathbf{L} commute with p^2 , so the commutator is

$$(0.4) \quad \frac{i}{\hbar} [H, L_x] = \frac{i}{\hbar} [V, yp_z - zp_y]$$

Although we also saw in the last post that \mathbf{L} commutes with V if the potential depends only on r , this doesn't help us here, since in order for the torque to be non-zero, the potential would have to have a non-radial dependence, since if $V = V(r)$, then $\mathbf{F} = -\nabla V = -\frac{dV}{dr} \hat{\mathbf{r}}$ and $\mathbf{N} = \mathbf{r} \times \left(-\frac{dV}{dr} \hat{\mathbf{r}}\right) = 0$.

As usual, we introduce an auxiliary function f to give the operators something to operate on:

$$(0.5) \quad \frac{i}{\hbar}[H, L_x]f = \frac{i}{\hbar}[V, yp_z - zp_y]f$$

$$(0.6) \quad = V \left(y \frac{\partial f}{\partial z} - z \frac{\partial f}{\partial y} \right) + z \frac{\partial(fV)}{\partial y} - y \frac{\partial(fV)}{\partial z}$$

$$(0.7) \quad = \left(z \frac{\partial V}{\partial y} - y \frac{\partial V}{\partial z} \right) f$$

Writing out the components of the torque $\mathbf{N} = \mathbf{r} \times (-\nabla V)$, we find that the last line above is equivalent to

$$(0.8) \quad \frac{i}{\hbar}[H, L_x]f = N_x f$$

Assuming there is no explicit time dependence in the system, this gives the result

$$(0.9) \quad \left\langle \frac{d\mathbf{L}}{dt} \right\rangle = \langle \mathbf{r} \times \mathbf{F} \rangle$$

$$(0.10) \quad = \langle \mathbf{N} \rangle$$