

## SPHERICAL HARMONIC AT THE TOP OF THE LADDER

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education Problem 4.22.

We can use the results obtained in expressing the angular momentum operators in spherical coordinates to get a general formula for the spherical harmonic  $Y_l^l$ . First, we note that since  $Y_l^l$  is the top spherical harmonic, applying the raising operator to it gives zero:  $L_+ Y_l^l = 0$ . We also have the raising operator expressed in terms of spherical coordinates:

$$(0.1) \quad L_+ = \hbar e^{i\phi} \left[ \frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} \right]$$

From this, we get a differential equation for  $Y_l^l$ . We will use the symbol  $f$  for the function since it simplifies the notation and allows us to use subscripts to represent partial derivatives.

$$(0.2) \quad f_\theta + i \cot \theta f_\phi = 0$$

$$(0.3) \quad \tan \theta f_\theta = -i f_\phi$$

At this point we can try separation of variables

$$(0.4) \quad f(\theta, \phi) = g(\theta)h(\phi)$$

Substituting this into the differential equation and dividing through by the product  $gh$  we get

$$(0.5) \quad \tan \theta \frac{g_\theta}{g} = -i \frac{h_\phi}{h}$$

Since the two sides of this equation depend on different independent variables, they must both be equal to the same constant, which we call  $l$ .

$$(0.6) \quad \tan \theta \frac{g_\theta}{g} = l$$

$$(0.7) \quad -i \frac{h_\phi}{h} = l$$

The first equation can be written as

$$(0.8) \quad \frac{dg}{g} = l \frac{\cos \theta}{\sin \theta} d\theta$$

Integrating, we get

$$(0.9) \quad \ln g = l \ln(\sin \theta) + \ln A$$

$$(0.10) \quad g(\theta) = A \sin^l \theta$$

Integrating the second equation gives

$$(0.11) \quad h(\phi) = B e^{il\phi}$$

so combining the two constants  $A$  and  $B$  into a single constant  $C$  we get the general form of  $Y_l^l$ :

$$(0.12) \quad Y_l^l(\theta, \phi) = C e^{il\phi} \sin^l \theta$$

We can determine  $C$  by normalization. The integral we need to evaluate is

$$(0.13) \quad \int_0^{2\pi} \int_0^\pi |Y_l^l(\theta, \phi)|^2 \sin \theta d\theta d\phi = 2\pi |C|^2 \int_0^\pi \sin^{2l+1} \theta d\theta$$

Integrals of this form give rise to recurrence relations.

$$(0.14)$$

$$\int_0^\pi \sin^{2l+1} \theta d\theta = \int_0^\pi (1 - \cos^2 \theta) \sin^{2l-1} \theta d\theta$$

$$(0.15) \quad = \int_0^\pi \sin^{2l-1} \theta d\theta - \frac{1}{2l} \cos \theta \sin^{2l} \theta \Big|_0^\pi - \frac{1}{2l} \int_0^\pi \sin^{2l+1} \theta d\theta$$

The integrated term is zero, and the last term contains the same integral we are trying to find, so we can collect terms to get the recurrence relation:

$$(0.16) \quad \int_0^\pi \sin^{2l+1} \theta d\theta = \frac{2l}{2l+1} \int_0^\pi \sin^{2l-1} \theta d\theta$$

For a given value of  $l$ , we need to iterate this formula until the power of the sine inside the integral is 1. Since the power drops by 2 on each iteration and we start with a power of  $2l+1$ , this will take  $l$  iterations, and we get

$$(0.17) \quad \int_0^\pi \sin^{2l+1} \theta d\theta = \frac{2l(2l-2)(2l-4)\dots 2}{(2l+1)(2l-1)(2l-3)\dots 3} \int_0^\pi \sin \theta d\theta$$

The term in the numerator is  $(2l)!! = 2^l l!$  where the  $!!$  symbol indicates a double factorial, which is the product of every second integer. The term in the denominator is  $(2l+1)!! = (2l+1)!/(2l)!! = (2l+1)!/(2^l l!)$ . The final integral is just 2. Thus we get

$$(0.18) \quad \int_0^\pi \sin^{2l+1} \theta d\theta = \frac{2(2^l l!)^2}{(2l+1)!}$$

Plugging this back into the original integral we get

$$(0.19) \quad \int_0^{2\pi} \int_0^\pi |Y_l^l(\theta, \phi)|^2 \sin \theta d\theta d\phi = \frac{4\pi(2^l l!)^2}{(2l+1)!} |C|^2$$

$$(0.20) \quad = 1$$

$$(0.21) \quad C = \sqrt{\frac{(2l+1)!}{4\pi} \frac{1}{2^l l!}}$$

This agrees with the result we got earlier, using associated Legendre functions.

#### PINGBACKS

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