

SPHERICAL HARMONIC AT THE TOP OF THE LADDER

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References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education Problem 4.22.

We can use the results obtained in expressing the angular momentum operators in spherical coordinates to get a general formula for the spherical harmonic Y_l^l . First, we note that since Y_l^l is the top spherical harmonic, applying the raising operator to it gives zero: $L_+ Y_l^l = 0$. We also have the raising operator expressed in terms of spherical coordinates:

$$L_+ = \hbar e^{i\phi} \left[\frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} \right] \quad (1)$$

From this, we get a differential equation for Y_l^l . We will use the symbol f for the function since it simplifies the notation and allows us to use subscripts to represent partial derivatives.

$$f_\theta + i \cot \theta f_\phi = 0 \quad (2)$$

$$\tan \theta f_\theta = -i f_\phi \quad (3)$$

At this point we can try separation of variables

$$f(\theta, \phi) = g(\theta)h(\phi) \quad (4)$$

Substituting this into the differential equation and dividing through by the product gh we get

$$\tan \theta \frac{g_\theta}{g} = -i \frac{h_\phi}{h} \quad (5)$$

Since the two sides of this equation depend on different independent variables, they must both be equal to the same constant, which we call l .

$$\tan \theta \frac{g_\theta}{g} = l \quad (6)$$

$$-i \frac{h_\phi}{h} = l \quad (7)$$

The first equation can be written as

$$\frac{dg}{g} = l \frac{\cos \theta}{\sin \theta} d\theta \quad (8)$$

Integrating, we get

$$\ln g = l \ln(\sin \theta) + \ln A \quad (9)$$

$$g(\theta) = A \sin^l \theta \quad (10)$$

Integrating the second equation gives

$$h(\phi) = B e^{il\phi} \quad (11)$$

so combining the two constants A and B into a single constant C we get the general form of Y_l^l :

$$Y_l^l(\theta, \phi) = C e^{il\phi} \sin^l \theta \quad (12)$$

We can determine C by normalization. The integral we need to evaluate is

$$\int_0^{2\pi} \int_0^\pi |Y_l^l(\theta, \phi)|^2 \sin \theta d\theta d\phi = 2\pi |C|^2 \int_0^\pi \sin^{2l+1} \theta d\theta \quad (13)$$

Integrals of this form give rise to recurrence relations.

$$\int_0^\pi \sin^{2l+1} \theta d\theta = \int_0^\pi (1 - \cos^2 \theta) \sin^{2l-1} \theta d\theta \quad (14)$$

$$= \int_0^\pi \sin^{2l-1} \theta d\theta - \frac{1}{2l} \cos \theta \sin^{2l} \theta \Big|_0^\pi - \frac{1}{2l} \int_0^\pi \sin^{2l+1} \theta d\theta \quad (15)$$

The integrated term is zero, and the last term contains the same integral we are trying to find, so we can collect terms to get the recurrence relation:

$$\int_0^\pi \sin^{2l+1} \theta d\theta = \frac{2l}{2l+1} \int_0^\pi \sin^{2l-1} \theta d\theta \quad (16)$$

For a given value of l , we need to iterate this formula until the power of the sine inside the integral is 1. Since the power drops by 2 on each iteration and we start with a power of $2l+1$, this will take l iterations, and we get

$$\int_0^\pi \sin^{2l+1} \theta d\theta = \frac{2l(2l-2)(2l-4)\dots 2}{(2l+1)(2l-1)(2l-3)\dots 3} \int_0^\pi \sin \theta d\theta \quad (17)$$

The term in the numerator is $(2l)!! = 2^l l!$ where the $!!$ symbol indicates a double factorial, which is the product of every second integer. The term in the denominator is $(2l+1)!! = (2l+1)!/(2l)!! = (2l+1)!/(2^l l!)$. The final integral is just 2. Thus we get

$$\int_0^\pi \sin^{2l+1} \theta d\theta = \frac{2(2^l l!)^2}{(2l+1)!} \quad (18)$$

Plugging this back into the original integral we get

$$\int_0^{2\pi} \int_0^\pi |Y_l^l(\theta, \phi)|^2 \sin \theta d\theta d\phi = \frac{4\pi(2^l l!)^2}{(2l+1)!} |C|^2 \quad (19)$$

$$= 1 \quad (20)$$

$$C = \sqrt{\frac{(2l+1)!}{4\pi}} \frac{1}{2^l l!} \quad (21)$$

This agrees with the result we got earlier, using associated Legendre functions.

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