SPHERICAL HARMONIC AT THE TOP OF THE LADDER

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education Problem 4.22.

We can use the results obtained in expressing the angular momentum operators in spherical coordinates to get a general formula for the spherical harmonic Y_l^l. First, we note that since Y_l^l is the top spherical harmonic, applying the raising operator to it gives zero: $L_+Y_l^l = 0$. We also have the raising operator expressed in terms of spherical coordinates:

$$L_+ = \hbar e^{i\phi}\left[\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right] \tag{1}$$

From this, we get a differential equation for Y_l^l. We will use the symbol f for the function since it simplifies the notation and allows us to use subscripts to represent partial derivatives.

$$\begin{aligned} f_\theta + i\cot\theta f_\phi &= 0 &(2)\\ \tan\theta f_\theta &= -if_\phi &(3)\end{aligned}$$

At this point we can try separation of variables

$$f(\theta,\phi) = g(\theta)h(\phi) \tag{4}$$

Substituting this into the differential equation and dividing through by the product gh we get

$$\tan\theta\frac{g_\theta}{g} = -i\frac{h_\phi}{h} \tag{5}$$

Since the two sides of this equation depend on different independent variables, they must both be equal to the same constant, which we call l.

$$\begin{aligned} \tan\theta\frac{g_\theta}{g} &= l &(6)\\ -i\frac{h_\phi}{h} &= l &(7)\end{aligned}$$

The first equation can be written as

$$\frac{dg}{g} = l\frac{\cos\theta}{\sin\theta}d\theta \tag{8}$$

Integrating, we get

$$\begin{aligned} \ln g &= l\ln(\sin\theta) + \ln A & (9) \\ g(\theta) &= A\sin^l\theta & (10) \end{aligned}$$

Integrating the second equation gives

$$h(\phi) = Be^{il\phi} \tag{11}$$

so combining the two constants A and B into a single constant C we get the general form of Y_l^l:

$$Y_l^l(\theta,\phi) = Ce^{il\phi}\sin^l\theta \tag{12}$$

We can determine C by normalization. The integral we need to evaluate is

$$\int_0^{2\pi}\int_0^{\pi}|Y_l^l(\theta,\phi)|^2\sin\theta d\theta d\phi = 2\pi|C|^2\int_0^{\pi}\sin^{2l+1}\theta d\theta \tag{13}$$

Integrals of this form give rise to recurrence relations.

$$\begin{aligned} \int_0^{\pi}\sin^{2l+1}\theta d\theta &= \int_0^{\pi}(1-\cos^2\theta)\sin^{2l-1}\theta d\theta & (14) \\ &= \int_0^{\pi}\sin^{2l-1}\theta d\theta - \frac{1}{2l}\cos\theta\sin^{2l}\theta\Big|_0^{\pi} - \frac{1}{2l}\int_0^{\pi}\sin^{2l+1}\theta d\theta & (15) \end{aligned}$$

The integrated term is zero, and the last term contains the same integral we are trying to find, so we can collect terms to get the recurrence relation:

$$\int_0^{\pi}\sin^{2l+1}\theta d\theta = \frac{2l}{2l+1}\int_0^{\pi}\sin^{2l-1}\theta d\theta \tag{16}$$

For a given value of l, we need to iterate this formula until the power of the sine inside the integral is 1. Since the power drops by 2 on each iteration and we start with a power of $2l+1$, this will take l iterations, and we get

$$\int_0^{\pi}\sin^{2l+1}\theta d\theta = \frac{2l(2l-2)(2l-4)\ldots 2}{(2l+1)(2l-1)(2l-3)\ldots 3}\int_0^{\pi}\sin\theta d\theta \tag{17}$$

The term in the numerator is $(2l)!! = 2^l l!$ where the !! symbol indicates a double factorial, which is the product of every second integer. The term in the denominator is $(2l+1)!! = (2l+1)!/(2l)!! = (2l+1)!/(2^l l!)$. The final integral is just 2. Thus we get

$$\int_0^\pi \sin^{2l+1}\theta d\theta = \frac{2(2^l l!)^2}{(2l+1)!} \tag{18}$$

Plugging this back into the original integral we get

$$\begin{aligned}
\int_0^{2\pi}\int_0^\pi |Y_l^l(\theta,\phi)|^2 \sin\theta d\theta d\phi &= \frac{4\pi(2^l l!)^2}{(2l+1)!}|C|^2 && (19)\\
&= 1 && (20)\\
C &= \sqrt{\frac{(2l+1)!}{4\pi}}\frac{1}{2^l l!} && (21)
\end{aligned}$$

This agrees with the result we got earlier, using associated Legendre functions.

PINGBACKS

Pingback: Spherical harmonics: normalization