

SPIN - STATISTICAL CALCULATIONS

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.27.

We saw in the last post that spin 1/2 is represented by spin operators that are 2×2 matrices. Here we present an example of how some of the standard statistical calculations are done using these matrices.

Suppose we have an electron in the spin state

$$(0.1) \quad \chi = A \begin{bmatrix} 3i \\ 4 \end{bmatrix}$$

This means that the electron's state is the sum of the spin up state multiplied by $3iA$ and the spin down state multiplied by $4A$. We can normalize the state by requiring its magnitude to be 1. That is

$$(0.2) \quad \chi^\dagger \chi = A^* A \begin{bmatrix} -3i & 4 \end{bmatrix} \begin{bmatrix} 3i \\ 4 \end{bmatrix} = 25 |A|^2 = 1$$

So $A = 1/5$.

We can find the mean values of the spin components by using an analogue of the method for calculating mean values with the spatial wave function. For an operator Q , instead of calculating $\int \Psi^* Q \Psi d^3 \mathbf{r}$, we calculate $\chi^\dagger Q \chi$. Using the spin matrices from the last post, we get

$$(0.3) \quad \langle S_x \rangle = \frac{\hbar}{50} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix} = 0$$

$$(0.4) \quad \langle S_y \rangle = \frac{\hbar}{50} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix} = -\frac{24\hbar}{50}$$

$$\langle S_z \rangle = \frac{\hbar}{50} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix} = -\frac{7\hbar}{50}$$

We can also get the standard deviations of the spin components. Since $S_x^2 = S_y^2 = S_z^2 = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\langle S_x^2 \rangle = \langle S_y^2 \rangle = \langle S_z^2 \rangle = (\hbar^2/4)|\chi|^2 = \hbar^2/4$. So

$$(0.5) \quad \sigma_{S_x} = (\langle S_x^2 \rangle - \langle S_x \rangle^2)^{1/2}$$

$$(0.6) \quad = \frac{\hbar}{2} = \frac{25\hbar}{50}$$

$$(0.7) \quad \sigma_{S_y} = (\langle S_y^2 \rangle - \langle S_y \rangle^2)^{1/2}$$

$$(0.8) \quad = \frac{7\hbar}{50}$$

$$(0.9) \quad \sigma_{S_z} = (\langle S_z^2 \rangle - \langle S_z \rangle^2)^{1/2}$$

$$(0.10) \quad = \frac{24\hbar}{50}$$

From the generalized uncertainty principle, we get

$$(0.11) \quad \sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2$$

The commutators for spin are:

$$(0.12) \quad [S_x, S_y] = i\hbar S_z$$

$$(0.13) \quad [S_y, S_z] = i\hbar S_x$$

$$(0.14) \quad [S_z, S_x] = i\hbar S_y$$

Working out the products, we get

$$(0.15) \quad \sigma_{S_x} \sigma_{S_y} = \frac{175}{2500} \hbar^2$$

$$(0.16) \quad = \frac{\hbar}{2} \frac{7\hbar}{50}$$

$$(0.17) \quad = \frac{\hbar}{2} |\langle S_z \rangle|$$

$$(0.18) \quad \sigma_{S_y} \sigma_{S_z} = \frac{168}{2500} \hbar^2$$

$$(0.19) \quad > \frac{\hbar}{2} |\langle S_x \rangle| = 0$$

$$(0.20) \quad \sigma_{S_z} \sigma_{S_x} = \frac{600}{2500} \hbar^2$$

$$(0.21) \quad = \frac{\hbar}{2} |\langle S_y \rangle|$$

so all relations are satisfied.