

SPIN - STATISTICAL CALCULATIONS

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.27.

We saw in the last post that spin 1/2 is represented by spin operators that are 2×2 matrices. Here we present an example of how some of the standard statistical calculations are done using these matrices.

Suppose we have an electron in the spin state

$$\chi = A \begin{bmatrix} 3i \\ 4 \end{bmatrix} \quad (1)$$

This means that the electron's state is the sum of the spin up state multiplied by $3iA$ and the spin down state multiplied by $4A$. We can normalize the state by requiring its magnitude to be 1. That is

$$\chi^\dagger \chi = A^* A \begin{bmatrix} -3i & 4 \end{bmatrix} \begin{bmatrix} 3i \\ 4 \end{bmatrix} = 25|A|^2 = 1 \quad (2)$$

So $A = 1/5$.

We can find the mean values of the spin components by using an analogue of the method for calculating mean values with the spatial wave function. For an operator Q , instead of calculating $\int \Psi^* Q \Psi d^3 \mathbf{r}$, we calculate $\chi^\dagger Q \chi$. Using the spin matrices from the last post, we get

$$\langle S_x \rangle = \frac{\hbar}{50} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix} = 0 \quad (3)$$

$$\langle S_y \rangle = \frac{\hbar}{50} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix} = -\frac{24\hbar}{50} \quad (4)$$

$$\langle S_z \rangle = \frac{\hbar}{50} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix} = -\frac{7\hbar}{50}$$

We can also get the standard deviations of the spin components. Since $S_x^2 = S_y^2 = S_z^2 = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\langle S_x^2 \rangle = \langle S_y^2 \rangle = \langle S_z^2 \rangle = (\hbar^2/4)|\chi|^2 = \hbar^2/4$.

So

$$\sigma_{S_x} = (\langle S_x^2 \rangle - \langle S_x \rangle^2)^{1/2} \quad (5)$$

$$= \frac{\hbar}{2} = \frac{25\hbar}{50} \quad (6)$$

$$\sigma_{S_y} = (\langle S_y^2 \rangle - \langle S_y \rangle^2)^{1/2} \quad (7)$$

$$= \frac{7\hbar}{50} \quad (8)$$

$$\sigma_{S_z} = (\langle S_z^2 \rangle - \langle S_z \rangle^2)^{1/2} \quad (9)$$

$$= \frac{24\hbar}{50} \quad (10)$$

From the generalized uncertainty principle, we get

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2 \quad (11)$$

The commutators for spin are:

$$[S_x, S_y] = i\hbar S_z \quad (12)$$

$$[S_y, S_z] = i\hbar S_x \quad (13)$$

$$[S_z, S_x] = i\hbar S_y \quad (14)$$

Working out the products, we get

$$\sigma_{S_x} \sigma_{S_y} = \frac{175}{2500} \hbar^2 \quad (15)$$

$$= \frac{\hbar}{2} \frac{7\hbar}{50} \quad (16)$$

$$= \frac{\hbar}{2} |\langle S_z \rangle| \quad (17)$$

$$\sigma_{S_y} \sigma_{S_z} = \frac{168}{2500} \hbar^2 \quad (18)$$

$$> \frac{\hbar}{2} |\langle S_x \rangle| = 0 \quad (19)$$

$$\sigma_{S_z} \sigma_{S_x} = \frac{600}{2500} \hbar^2 \quad (20)$$

$$= \frac{\hbar}{2} |\langle S_y \rangle| \quad (21)$$

so all relations are satisfied.