SPIN: THE X AND Y COMPONENTS

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.29.

Just as we found the eigenvalues and eigenspinors of S_z, we can do the same for the other 2 components. For S_y, for example, we have

$$S_y = \frac{\hbar}{2}\left[\begin{array}{cc} 0 & -i \\ i & 0 \end{array}\right] \tag{1}$$

We have, first, for the eigenvalues

$$\left|\begin{array}{cc} -\lambda & -i\frac{\hbar}{2} \\ i\frac{\hbar}{2} & -\lambda \end{array}\right| = \lambda^2 - \frac{\hbar^2}{4} = 0 \tag{2}$$

from which

$$\lambda = \pm\frac{\hbar}{2} \tag{3}$$

So the eigenvalues of S_y are also $\pm\hbar/2$.

Solving the equation to find the eigenspinors gives

$$\frac{\hbar}{2}\left(\begin{array}{cc} 0 & -i \\ i & 0 \end{array}\right)\left(\begin{array}{c} \alpha \\ \beta \end{array}\right) = \pm\frac{\hbar}{2}\left(\begin{array}{c} \alpha \\ \beta \end{array}\right) \tag{4}$$

$$\chi_+^{(y)} = \frac{1}{\sqrt{2}}\left(\begin{array}{c} 1 \\ i \end{array}\right) \tag{5}$$

$$\chi_-^{(y)} = \frac{1}{\sqrt{2}}\left(\begin{array}{c} 1 \\ -i \end{array}\right) \tag{6}$$

A similar calculation for S_x gives

$$\chi_+^{(x)} = \frac{1}{\sqrt{2}}\left(\begin{array}{c} 1 \\ 1 \end{array}\right) \tag{7}$$

$$\chi_-^{(x)} = \frac{1}{\sqrt{2}}\left(\begin{array}{c} 1 \\ -1 \end{array}\right) \tag{8}$$

To find the probabilities of a particle that is in a general spin state $\chi = \begin{pmatrix} a \\ b \end{pmatrix}$ being in each of the eigenstates, we need to express the general state χ in terms of $\chi_+^{(y)}$ and $\chi_-^{(y)}$. First we convert the basis vectors:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(\chi_+^{(y)} + \chi_-^{(y)}) \tag{9}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{-i}{\sqrt{2}}(\chi_+^{(y)} - \chi_-^{(y)}) \tag{10}$$

so

$$\chi = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(a-ib)\chi_+^{(y)} + \frac{1}{\sqrt{2}}(a+ib)\chi_-^{(y)} \tag{11}$$

Thus the probability of the particle being in state $\chi_+^{(y)}$ is $|a-ib|^2/2$ and in state $\chi_-^{(y)}$ is $|a+ib|^2/2$.

The sum of these probabilities is (remember that a and b are, in general, complex, so we can't just say $|a+ib|^2 = a^2+b^2$)

$$\frac{|a-ib|^2}{2} + \frac{|a+ib|^2}{2} = \frac{1}{2}(|a|^2+|b|^2 - ia^*b + iab^* + |a|^2+|b|^2 + ia^*b - iab^*) \tag{12}$$

$$= \frac{1}{2}(2|a|^2+2|b|^2) \tag{13}$$

$$= 1 \tag{14}$$

since $|a|^2+|b|^2 = 1$ from normalization.

Since $S_x^2 = S_y^2 = S_z^2 = \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, measuring S_y^2 will always give $\hbar^2/4$. Since the eigenvalues of S_y are $\pm\hbar/2$ this follows.

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