SPIN: THE X AND Y COMPONENTS

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.29.

Just as we found the eigenvalues and eigenspinors of S_z, we can do the same for the other 2 components. For S_y, for example, we have

$$S_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{1}$$

We have, first, for the eigenvalues

$$\begin{vmatrix} -\lambda & -i\frac{\hbar}{2} \\ i\frac{\hbar}{2} & -\lambda \end{vmatrix} = \lambda^2 - \frac{\hbar^2}{4} = 0 \tag{2}$$

from which

$$\lambda = \pm\frac{\hbar}{2} \tag{3}$$

So the eigenvalues of S_y are also $\pm\hbar/2$.

Solving the equation to find the eigenspinors gives

$$\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \pm\frac{\hbar}{2}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{4}$$

$$\chi_+^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \tag{5}$$

$$\chi_-^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} \tag{6}$$

A similar calculation for S_x gives

$$\chi_+^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{7}$$

$$\chi_{-}^{(x)}=\frac{1}{\sqrt{2}}\left(\begin{array}{c}1\\-1\end{array}\right) \tag{8}$$

To find the probabilities of a particle that is in a general spin state $\chi=\left(\begin{array}{c}a\\b\end{array}\right)$ being in each of the eigenstates, we need to express the general state χ in terms of $\chi_{+}^{(y)}$ and $\chi_{-}^{(y)}$. First we convert the basis vectors:

$$\left(\begin{array}{c}1\\0\end{array}\right)=\frac{1}{\sqrt{2}}(\chi_{+}^{(y)}+\chi_{-}^{(y)}) \tag{9}$$

$$\left(\begin{array}{c}0\\1\end{array}\right)=\frac{-i}{\sqrt{2}}(\chi_{+}^{(y)}-\chi_{-}^{(y)}) \tag{10}$$

so

$$\chi=a\left(\begin{array}{c}1\\0\end{array}\right)+b\left(\begin{array}{c}0\\1\end{array}\right)=\frac{1}{\sqrt{2}}(a-ib)\chi_{+}^{(y)}+\frac{1}{\sqrt{2}}(a+ib)\chi_{-}^{(y)} \tag{11}$$

Thus the probability of the particle being in state $\chi_{+}^{(y)}$ is $|a-ib|^{2}/2$ and in state $\chi_{-}^{(y)}$ is $|a+ib|^{2}/2$.

The sum of these probabilities is (remember that a and b are, in general, complex, so we can't just say $|a+ib|^{2}=a^{2}+b^{2}$)

$$\frac{|a-ib|^{2}}{2}+\frac{|a+ib|^{2}}{2}=\frac{1}{2}(|a|^{2}+|b|^{2}-ia^{*}b+iab^{*}+|a|^{2}+|b|^{2}+ia^{*}b-iab^{*}) \tag{12}$$

$$=\frac{1}{2}(2|a|^{2}+2|b|^{2}) \tag{13}$$

$$=1 \tag{14}$$

since $|a|^{2}+|b|^{2}=1$ from normalization.

Since $S_{x}^{2}=S_{y}^{2}=S_{z}^{2}=\frac{\hbar^{2}}{4}\left(\begin{array}{cc}1&0\\0&1\end{array}\right)$, measuring S_{y}^{2} will always give $\hbar^{2}/4$. Since the eigenvalues of S_{y} are $\pm\hbar/2$ this follows.

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