

SPIN 1/2 ALONG AN ARBITRARY DIRECTION

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.30.

Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 14, Exercise 14.3.2.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

We've seen what the spin 1/2 matrices look like along the 3 rectangular coordinate axes. From this, we can derive an expression for the spin component along an arbitrary direction $\hat{\mathbf{r}}$. The unit radius vector is

$$\hat{\mathbf{r}} = \sin \theta \cos \phi \hat{\mathbf{i}} + \sin \theta \sin \phi \hat{\mathbf{j}} + \cos \theta \hat{\mathbf{k}} \quad (1)$$

We can get S_r by combining S_x , S_y and S_z according to the formula for the radius vector:

$$S_r = \mathbf{S} \cdot \hat{\mathbf{r}} \quad (2)$$

$$= \sin \theta \cos \phi S_x + \sin \theta \sin \phi S_y + \cos \theta S_z \quad (3)$$

By using the forms for the matrices derived earlier, and $\cos \phi \pm i \sin \phi = e^{\pm i\phi}$ we get

$$S_r = \frac{\hbar}{2} \begin{pmatrix} \cos \theta & \sin \theta e^{-i\phi} \\ \sin \theta e^{i\phi} & -\cos \theta \end{pmatrix} \quad (4)$$

The eigenvalues of this matrix are calculated in the usual way

$$\begin{vmatrix} \frac{\hbar}{2} \cos \theta - \lambda & \frac{\hbar}{2} \sin \theta e^{-i\phi} \\ \frac{\hbar}{2} \sin \theta e^{i\phi} & -\frac{\hbar}{2} \cos \theta - \lambda \end{vmatrix} = -\frac{\hbar^2}{4} [\cos^2 \theta + \sin^2 \theta] + \lambda^2 = 0 \quad (5)$$

We get $\lambda = \pm \hbar/2$ as before so all is well at this stage.

To get the eigenspinors, we must solve

$$\frac{\hbar}{2} \begin{pmatrix} \cos \theta \pm 1 & \sin \theta e^{-i\phi} \\ \sin \theta e^{i\phi} & -\cos \theta \pm 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \quad (6)$$

We get the equations

$$(\cos \theta \pm 1)\alpha + \sin \theta e^{-i\phi} \beta = 0 \quad (7)$$

$$\sin \theta e^{i\phi} \alpha - (-\cos \theta \pm 1)\beta = 0 \quad (8)$$

The two solutions (one for each sign) are

$$\beta_+ = -e^{i\phi} \frac{\cos \theta - 1}{\sin \theta} \alpha_+ \quad (9)$$

$$\beta_- = -e^{i\phi} \frac{\cos \theta + 1}{\sin \theta} \alpha_- \quad (10)$$

We can use the double-angle trig identities to simplify these expressions:

$$\sin \theta = 2 \sin(\theta/2) \cos(\theta/2) \quad (11)$$

$$\cos \theta = \cos^2(\theta/2) - \sin^2(\theta/2) \quad (12)$$

Substituting these together with $\cos^2(\theta/2) + \sin^2(\theta/2) = 1$ and simplifying leads to

$$\beta_+ = e^{i\phi} \frac{\sin(\theta/2)}{\cos(\theta/2)} \alpha_+ \quad (13)$$

$$\beta_- = -e^{i\phi} \frac{\cos(\theta/2)}{\sin(\theta/2)} \alpha_- \quad (14)$$

The eigenspinors should be normalized, so

$$|\beta_+|^2 + |\alpha_+|^2 = |\alpha_+|^2 \left(\frac{\sin^2(\theta/2)}{\cos^2(\theta/2)} + 1 \right) \quad (15)$$

$$= |\alpha_+|^2 \left(\frac{\sin^2(\theta/2) + \cos^2(\theta/2)}{\cos^2(\theta/2)} \right) \quad (16)$$

$$= \frac{|\alpha_+|^2}{\cos^2(\theta/2)} \quad (17)$$

$$= 1 \quad (18)$$

Thus we can take

$$\alpha_+ = \cos \frac{\theta}{2} \quad (19)$$

Other answers are possible, since we can multiply α_+ by any complex exponential, as all that is important is that its magnitude is 1.

A similar calculation for the other solution leads to

$$\frac{|\alpha_-|^2}{\sin^2(\theta/2)} = 1 \quad (20)$$

We can take

$$\alpha_- = \sin \frac{\theta}{2} \quad (21)$$

These choices lead to

$$\chi_+^{(r)} = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi} \sin(\theta/2) \end{pmatrix} \quad (22)$$

$$\chi_-^{(r)} = \begin{pmatrix} \sin(\theta/2) \\ -e^{i\phi} \cos(\theta/2) \end{pmatrix} \quad (23)$$

If we want the answer in Griffiths, we would choose $\alpha_- = e^{-i\phi} \sin \frac{\theta}{2}$, which leads to the answer:

$$\chi_+^{(r)} = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi} \sin(\theta/2) \end{pmatrix} \quad (24)$$

$$\chi_-^{(r)} = \begin{pmatrix} e^{-i\phi} \sin(\theta/2) \\ -\cos(\theta/2) \end{pmatrix} \quad (25)$$

The phase difference between the two components is the same in each solution.

Shankar's equations 14.3.28 use a slightly different phase, giving

$$|\hat{n}+\rangle = \begin{bmatrix} \cos \frac{\theta}{2} e^{-i\phi/2} \\ \sin \frac{\theta}{2} e^{i\phi/2} \end{bmatrix} \quad (26)$$

$$|\hat{n}-\rangle = \begin{bmatrix} -\sin \frac{\theta}{2} e^{-i\phi/2} \\ \cos \frac{\theta}{2} e^{i\phi/2} \end{bmatrix} \quad (27)$$

We can calculate $\langle \mathbf{S} \rangle$ by using the spin matrices

$$S_x = \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (28)$$

$$S_y = \frac{\hbar}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad (29)$$

$$S_z = \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (30)$$

We have

$$\langle \hat{n} + |S_x| \hat{n} + \rangle = \frac{\hbar}{2} \begin{bmatrix} \cos \frac{\theta}{2} e^{i\phi/2} & \sin \frac{\theta}{2} e^{-i\phi/2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos \frac{\theta}{2} e^{-i\phi/2} \\ \sin \frac{\theta}{2} e^{i\phi/2} \end{bmatrix} \quad (31)$$

$$= \frac{\hbar}{2} \begin{bmatrix} \cos \frac{\theta}{2} e^{i\phi/2} & \sin \frac{\theta}{2} e^{-i\phi/2} \end{bmatrix} \begin{bmatrix} \sin \frac{\theta}{2} e^{i\phi/2} \\ \cos \frac{\theta}{2} e^{-i\phi/2} \end{bmatrix} \quad (32)$$

$$= \frac{\hbar}{2} \sin \frac{\theta}{2} \cos \frac{\theta}{2} (e^{i\phi} + e^{-i\phi}) \quad (33)$$

$$= \frac{\hbar}{2} \sin \theta \cos \phi \quad (34)$$

$$\langle \hat{n} + |S_y| \hat{n} + \rangle = \frac{\hbar}{2} \begin{bmatrix} \cos \frac{\theta}{2} e^{i\phi/2} & \sin \frac{\theta}{2} e^{-i\phi/2} \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} \cos \frac{\theta}{2} e^{-i\phi/2} \\ \sin \frac{\theta}{2} e^{i\phi/2} \end{bmatrix} \quad (35)$$

$$= \frac{i\hbar}{2} \begin{bmatrix} \cos \frac{\theta}{2} e^{i\phi/2} & \sin \frac{\theta}{2} e^{-i\phi/2} \end{bmatrix} \begin{bmatrix} -\sin \frac{\theta}{2} e^{i\phi/2} \\ \cos \frac{\theta}{2} e^{-i\phi/2} \end{bmatrix} \quad (36)$$

$$= -\frac{\hbar}{2i} \sin \frac{\theta}{2} \cos \frac{\theta}{2} (-e^{i\phi} + e^{-i\phi}) \quad (37)$$

$$= \frac{\hbar}{2} \sin \theta \sin \phi \quad (38)$$

$$\langle \hat{n} + |S_z| \hat{n} + \rangle = \frac{\hbar}{2} \begin{bmatrix} \cos \frac{\theta}{2} e^{i\phi/2} & \sin \frac{\theta}{2} e^{-i\phi/2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos \frac{\theta}{2} e^{-i\phi/2} \\ \sin \frac{\theta}{2} e^{i\phi/2} \end{bmatrix} \quad (39)$$

$$= \frac{\hbar}{2} \begin{bmatrix} \cos \frac{\theta}{2} e^{i\phi/2} & \sin \frac{\theta}{2} e^{-i\phi/2} \end{bmatrix} \begin{bmatrix} \cos \frac{\theta}{2} e^{-i\phi/2} \\ -\sin \frac{\theta}{2} e^{i\phi/2} \end{bmatrix} \quad (40)$$

$$= \frac{\hbar}{2} \left(\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} \right) \quad (41)$$

$$= \frac{\hbar}{2} \cos \theta \quad (42)$$

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