SPIN 1/2 ALONG AN ARBITRARY DIRECTION

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.30.

Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press. Chapter 14, Exercise 14.3.2.

[If some equations are too small to read easily, use your browser's magnifying option (Ctrl + on Chrome, probably something similar on other browsers).]

We've seen what the spin 1/2 matrices look like along the 3 rectangular coordinate axes. From this, we can derive an expression for the spin component along an arbitrary direction $\hat{\mathbf{r}}$. The unit radius vector is

$$\hat{\mathbf{r}}=\sin\theta\cos\phi\hat{\mathbf{i}}+\sin\theta\sin\phi\hat{\mathbf{j}}+\cos\theta\hat{\mathbf{k}} \tag{0.1}$$

We can get S_r by combining S_x, S_y and S_z according to the formula for the radius vector:

$$\begin{aligned}\mathsf{S}_r &= \mathsf{S}\cdot\hat{\mathbf{r}} &(0.2)\\ &= \sin\theta\cos\phi S_x+\sin\theta\sin\phi S_y+\cos\theta S_z &(0.3)\end{aligned}$$

By using the forms for the matrices derived earlier, and $\cos\phi\pm i\sin\phi=e^{\pm i\phi}$ we get

$$\mathsf{S}_r=\frac{\hbar}{2}\left(\begin{array}{cc}\cos\theta & \sin\theta e^{-i\phi}\\ \sin\theta e^{i\phi} & -\cos\theta\end{array}\right) \tag{0.4}$$

The eigenvalues of this matrix are calculated in the usual way

$$\left|\begin{array}{cc}\frac{\hbar}{2}\cos\theta-\lambda & \frac{\hbar}{2}\sin\theta e^{-i\phi}\\ \frac{\hbar}{2}\sin\theta e^{i\phi} & -\frac{\hbar}{2}\cos\theta-\lambda\end{array}\right|=-\frac{\hbar^2}{4}\left[\cos^2\theta+\sin^2\theta\right]+\lambda^2=0 \tag{0.5}$$

We get $\lambda=\pm\hbar/2$ as before so all is well at this stage.

To get the eigenspinors, we must solve

$$\frac{\hbar}{2}\left(\begin{array}{cc}\cos\theta\pm1 & \sin\theta e^{-i\phi}\\ \sin\theta e^{i\phi} & -\cos\theta\pm1\end{array}\right)\left(\begin{array}{c}\alpha\\ \beta\end{array}\right)=0 \tag{0.6}$$

We get the equations

$$(\cos\theta \pm 1)\alpha + \sin\theta e^{-i\phi}\beta = 0 \tag{0.7}$$

$$\sin\theta e^{i\phi}\alpha - (-\cos\theta \pm 1)\beta = 0 \tag{0.8}$$

The two solutions (one for each sign) are

$$\beta_+ = -e^{i\phi}\frac{\cos\theta - 1}{\sin\theta}\alpha_+ \tag{0.9}$$

$$\beta_- = -e^{i\phi}\frac{\cos\theta + 1}{\sin\theta}\alpha_- \tag{0.10}$$

We can use the double-angle trig identities to simplify these expressions:

$$\sin\theta = 2\sin(\theta/2)\cos(\theta/2) \tag{0.11}$$

$$\cos\theta = \cos^2(\theta/2) - \sin^2(\theta/2) \tag{0.12}$$

Substituting these together with $\cos^2(\theta/2) + \sin^2(\theta/2) = 1$ and simplifying leads to

$$\beta_+ = e^{i\phi}\frac{\sin(\theta/2)}{\cos(\theta/2)}\alpha_+ \tag{0.13}$$

$$\beta_- = -e^{i\phi}\frac{\cos(\theta/2)}{\sin(\theta/2)}\alpha_- \tag{0.14}$$

The eigenspinors should be normalized, so

$$|\beta_+|^2 + |\alpha_+|^2 = |\alpha_+|^2\left(\frac{\sin^2(\theta/2)}{\cos^2(\theta/2)} + 1\right) \tag{0.15}$$

$$= |\alpha_+|^2\left(\frac{\sin^2(\theta/2) + \cos^2(\theta/2)}{\cos^2(\theta/2)}\right) \tag{0.16}$$

$$= \frac{|\alpha_+|^2}{\cos^2(\theta/2)} \tag{0.17}$$

$$= 1 \tag{0.18}$$

Thus we can take

$$\alpha_+ = \cos\frac{\theta}{2} \tag{0.19}$$

Other answers are possible, since we can multiply α_+ by any complex exponential, as all that is important is that its magnitude is 1.

A similar calculation for the other solution leads to

$$\frac{|\alpha_-|^2}{\sin^2(\theta/2)} = 1 \tag{0.20}$$

We can take

$$\alpha_- = \sin\frac{\theta}{2} \tag{0.21}$$

These choices lead to

$$\chi_+^{(r)} = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix} \tag{0.22}$$

$$\chi_-^{(r)} = \begin{pmatrix} \sin(\theta/2) \\ -e^{i\phi}\cos(\theta/2) \end{pmatrix} \tag{0.23}$$

If we want the answer in Griffiths, we would choose $\alpha_- = e^{-i\phi}\sin\frac{\theta}{2}$, which leads to the answer:

$$\chi_+^{(r)} = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix} \tag{0.24}$$

$$\chi_-^{(r)} = \begin{pmatrix} e^{-i\phi}\sin(\theta/2) \\ -\cos(\theta/2) \end{pmatrix} \tag{0.25}$$

The phase difference between the two components is the same in each solution.

Shankar's equations 14.3.28 use a slightly different phase, giving

$$|\hat{n}+\rangle = \begin{bmatrix} \cos\frac{\theta}{2}e^{-i\phi/2} \\ \sin\frac{\theta}{2}e^{i\phi/2} \end{bmatrix} \tag{0.26}$$

$$|\hat{n}-\rangle = \begin{bmatrix} -\sin\frac{\theta}{2}e^{-i\phi/2} \\ \cos\frac{\theta}{2}e^{i\phi/2} \end{bmatrix} \tag{0.27}$$

We can calculate $\langle\mathbf{S}\rangle$ by using the spin matrices

$$S_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{0.28}$$

$$S_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{0.29}$$

$$S_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{0.30}$$

We have

$$\begin{aligned}
\langle \hat{n}+|S_x|\hat{n}+\rangle &= \frac{\hbar}{2}\begin{bmatrix} \cos\frac{\theta}{2}e^{i\phi/2} & \sin\frac{\theta}{2}e^{-i\phi/2} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \cos\frac{\theta}{2}e^{-i\phi/2} \\ \sin\frac{\theta}{2}e^{i\phi/2} \end{bmatrix} && (0.31)\\
&= \frac{\hbar}{2}\begin{bmatrix} \cos\frac{\theta}{2}e^{i\phi/2} & \sin\frac{\theta}{2}e^{-i\phi/2} \end{bmatrix}\begin{bmatrix} \sin\frac{\theta}{2}e^{i\phi/2} \\ \cos\frac{\theta}{2}e^{-i\phi/2} \end{bmatrix} && (0.32)\\
&= \frac{\hbar}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\left(e^{i\phi}+e^{-i\phi}\right) && (0.33)\\
&= \frac{\hbar}{2}\sin\theta\cos\phi && (0.34)\\
\left\langle \hat{n}+|S_y|\hat{n}+\right\rangle &= \frac{\hbar}{2}\begin{bmatrix} \cos\frac{\theta}{2}e^{i\phi/2} & \sin\frac{\theta}{2}e^{-i\phi/2} \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} \cos\frac{\theta}{2}e^{-i\phi/2} \\ \sin\frac{\theta}{2}e^{i\phi/2} \end{bmatrix} && (0.35)\\
&= \frac{i\hbar}{2}\begin{bmatrix} \cos\frac{\theta}{2}e^{i\phi/2} & \sin\frac{\theta}{2}e^{-i\phi/2} \end{bmatrix}\begin{bmatrix} -\sin\frac{\theta}{2}e^{i\phi/2} \\ \cos\frac{\theta}{2}e^{-i\phi/2} \end{bmatrix} && (0.36)\\
&= -\frac{\hbar}{2i}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\left(-e^{i\phi}+e^{-i\phi}\right) && (0.37)\\
&= \frac{\hbar}{2}\sin\theta\sin\phi && (0.38)\\
\langle \hat{n}+|S_z|\hat{n}+\rangle &= \frac{\hbar}{2}\begin{bmatrix} \cos\frac{\theta}{2}e^{i\phi/2} & \sin\frac{\theta}{2}e^{-i\phi/2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \cos\frac{\theta}{2}e^{-i\phi/2} \\ \sin\frac{\theta}{2}e^{i\phi/2} \end{bmatrix} && (0.39)\\
&= \frac{\hbar}{2}\begin{bmatrix} \cos\frac{\theta}{2}e^{i\phi/2} & \sin\frac{\theta}{2}e^{-i\phi/2} \end{bmatrix}\begin{bmatrix} \cos\frac{\theta}{2}e^{-i\phi/2} \\ -\sin\frac{\theta}{2}e^{i\phi/2} \end{bmatrix} && (0.40)\\
&= \frac{\hbar}{2}\left(\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\right) && (0.41)\\
&= \frac{\hbar}{2}\cos\theta && (0.42)
\end{aligned}$$

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