

SPIN 1/2 PARTICLE IN A MAGNETIC FIELD

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Section 4.4.2 & Problem 4.32.

In classical electromagnetic theory, the motion of charge (as in the current in a wire) gives rise to a magnetic field. In particular, if we have a charged object that is spinning about some axis, the motion of the charge gives the object a magnetic dipole moment μ which is related to the spin angular momentum \mathbf{S} by the *gyromagnetic ratio* γ :

$$(0.1) \quad \mu = \gamma \mathbf{S}$$

The value of γ depends on the charge, shape and mass of the object. The interaction energy between a magnetic dipole and a magnetic field \mathbf{B} is:

$$(0.2) \quad H = -\mu \cdot \mathbf{B}$$

so for a spinning charged particle whose centre of mass is at rest, we get

$$(0.3) \quad H = -\gamma \mathbf{S} \cdot \mathbf{B}$$

All this is classical (non-quantum) physics, but as usual, we can translate the equations into quantum theory in a straightforward way. If we take the spin to be a matrix, and consider spin 1/2, then

$$(0.4) \quad H = -\gamma \mathbf{B} \cdot \mathbf{S}$$

If we take the magnetic field to be a constant and aligned in the z direction so that $\mathbf{B} = B_0 \hat{\mathbf{k}}$, then

$$(0.5) \quad H = -\gamma B_0 \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Since the energy is just a constant multiplied by S_z , the eigenspinors are the same as those of S_z and the corresponding energies are

$$(0.6) \quad E_{\mp} = \pm \gamma B_0 \frac{\hbar}{2}$$

where E_- corresponds to the energy is the spin down state and E_+ to spin up.

We can write the time-dependent solution in the same way as for the spatial wave function: we multiply each stationary state by $e^{-iEt/\hbar}$ and add up all the terms to get the general solution. In the example here, the general state is

$$(0.7) \quad \chi(t) = a\chi_+ e^{i\gamma B_0 t/2} + b\chi_- e^{-i\gamma B_0 t/2} = \begin{bmatrix} a e^{i\gamma B_0 t/2} \\ b e^{-i\gamma B_0 t/2} \end{bmatrix}$$

The constants a and b are determined from the initial conditions and from normalization, so they have to satisfy $|a|^2 + |b|^2 = 1$. The most general solution is then

$$(0.8) \quad \chi(t) = \begin{bmatrix} A e^{i\beta} e^{i\gamma B_0 t/2} \\ B e^{i\delta} e^{-i\gamma B_0 t/2} \end{bmatrix}$$

where A , B , β and δ are real constants determined by the initial conditions. From the normalization condition, we must have $A^2 + B^2 = 1$, so we might as well define

$$(0.9) \quad A = \cos \frac{\alpha}{2}$$

$$(0.10) \quad B = \sin \frac{\alpha}{2}$$

(The reason for using half angles is that α is an angle that turns up in a discussion of Larmor precession, which we won't get into here.)

If we take $\beta = \delta = 0$, then we can consider the case

$$(0.11) \quad \chi(t) = \begin{bmatrix} \cos(\alpha/2) e^{i\gamma B_0 t/2} \\ \sin(\alpha/2) e^{-i\gamma B_0 t/2} \end{bmatrix}$$

From this, we can work out the probabilities of getting various values for each of the spin components. We need to express $\chi(t)$ in terms of the eigenspinors of S_x :

$$(0.12) \quad \begin{pmatrix} \cos(\alpha/2) e^{i\gamma B_0 t/2} \\ \sin(\alpha/2) e^{-i\gamma B_0 t/2} \end{pmatrix} = \frac{a}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{b}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

This gives us a set of simultaneous equations in a and b which can be solved to give:

$$(0.13) \quad a = \frac{\sqrt{2}}{2} \left(\cos(\alpha/2) e^{i\gamma B_0 t/2} + \sin(\alpha/2) e^{-i\gamma B_0 t/2} \right)$$

$$(0.14) \quad b = \frac{\sqrt{2}}{2} \left(\cos(\alpha/2) e^{i\gamma B_0 t/2} - \sin(\alpha/2) e^{-i\gamma B_0 t/2} \right)$$

The probability of finding S_x in state $\hbar/2$ is therefore $|a|^2$ which works out to

$$(0.15)$$

$$|a|^2 = \frac{1}{2} (\cos(\alpha/2) e^{-i\gamma B_0 t/2} + \sin(\alpha/2) e^{i\gamma B_0 t/2}) (\cos(\alpha/2) e^{i\gamma B_0 t/2} + \sin(\alpha/2) e^{-i\gamma B_0 t/2})$$

$$(0.16)$$

$$= \frac{1}{2} (1 + \sin(\alpha/2) \cos(\alpha/2) (e^{i\gamma B_0 t} + e^{-i\gamma B_0 t}))$$

$$(0.17)$$

$$= \frac{1}{2} (1 + 2 \sin(\alpha/2) \cos(\alpha/2) \cos(\gamma B_0 t))$$

$$(0.18)$$

$$= \frac{1}{2} (1 + \sin \alpha \cos(\gamma B_0 t))$$

For completeness, the probability of getting $-\hbar/2$ is

$$(0.19) \quad |b|^2 = \frac{1}{2} (1 - \sin \alpha \cos(\gamma B_0 t))$$

For S_y we can use the eigenspinors to get

$$(0.20) \quad \begin{pmatrix} \cos(\alpha/2) e^{i\gamma B_0 t/2} \\ \sin(\alpha/2) e^{-i\gamma B_0 t/2} \end{pmatrix} = \frac{a}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{b}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$(0.21) \quad a = \frac{\sqrt{2}}{2} \left(\cos(\alpha/2) e^{i\gamma B_0 t/2} - i \sin(\alpha/2) e^{-i\gamma B_0 t/2} \right)$$

$$(0.22) \quad b = \frac{\sqrt{2}}{2} \left(\cos(\alpha/2) e^{i\gamma B_0 t/2} + i \sin(\alpha/2) e^{-i\gamma B_0 t/2} \right)$$

The probability of finding S_y in state $\hbar/2$ is therefore $|a|^2$ which works out to

$$(0.23) \quad |a|^2 = \frac{1}{2}(1 - \sin \alpha \sin(\gamma B_0 t))$$

For S_z we have

$$(0.24) \quad \begin{pmatrix} \cos(\alpha/2)e^{i\gamma B_0 t/2} \\ \sin(\alpha/2)e^{-i\gamma B_0 t/2} \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

so the probability of measuring S_z as $\hbar/2$ is

$$(0.25) \quad |a|^2 = \cos^2(\alpha/2)$$

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