

SPIN 1/2 PARTICLE IN TIME-VARYING MAGNETIC FIELD

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.33.

Here's an example of a time-dependent Hamiltonian in a spin 1/2 system. We saw in the last post that the energy of a particle with spin in a magnetic field is

$$(1) \quad H = -\gamma \mathbf{B} \cdot \mathbf{S}$$

Suppose now that the magnetic field is parallel to the z axis, but oscillates, so it is given by

$$(2) \quad \mathbf{B} = B_0 \cos(\omega t) \hat{\mathbf{k}}$$

The Hamiltonian for a spin 1/2 particle such as an electron is therefore $H = -\gamma B_0 \cos(\omega t) S_z$, or:

$$(3) \quad H = -\frac{\gamma B_0 \hbar}{2} \cos \omega t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

To find $\chi(t)$ we need to solve the time-dependent Schrödinger equation

$$(4) \quad i\hbar \frac{\partial \chi}{\partial t} = H \chi$$

If $\chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$, we get two differential equations

$$(5) \quad \frac{\partial \chi_{1,2}}{\partial t} = \pm i \frac{\gamma B_0}{2} (\cos \omega t) \chi_{1,2}$$

$$(6) \quad \frac{d\chi_{1,2}}{\chi_{1,2}} = \pm i \frac{\gamma B_0}{2} (\cos \omega t) dt$$

$$(7) \quad \ln \chi_{1,2} = \pm i \frac{\gamma B_0}{2\omega} (\sin \omega t) + \ln C_{1,2}$$

$$(8) \quad \chi_{1,2}(t) = C_{1,2} e^{\pm i \gamma B_0 (\sin \omega t) / 2\omega}$$

where $C_{1,2}$ are constants that must be found from the initial condition. If

we take the initial state of the electron to be spin up, then $\chi(0) = \chi_+^{(x)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. From this condition we find $C_1 = C_2 = 1/\sqrt{2}$, so the final solution is:

$$(9) \quad \chi(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\gamma B_0(\sin \omega t)/2\omega} \\ e^{-i\gamma B_0(\sin \omega t)/2\omega} \end{pmatrix}$$

To find the probability of getting $\hbar/2$ if we measure S_x , we can use the same method as in the last post, so we must express $\chi(t)$ in terms of the eigenspinors of S_x :

$$(10) \quad \chi(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\gamma B_0(\sin \omega t)/2\omega} \\ e^{-i\gamma B_0(\sin \omega t)/2\omega} \end{pmatrix} = \frac{a}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{b}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

from which we can solve for a and b :

$$(11) \quad a + b = e^{i\gamma B_0(\sin \omega t)/2\omega}$$

$$(12) \quad a - b = e^{-i\gamma B_0(\sin \omega t)/2\omega}$$

$$(13) \quad a = \cos \left(\frac{\gamma B_0}{2\omega} (\sin \omega t) \right)$$

$$(14) \quad b = i \sin \left(\frac{\gamma B_0}{2\omega} (\sin \omega t) \right)$$

The probability of a measurement of S_x giving $-\hbar/2$ is therefore $|b|^2 = \sin^2 \left(\frac{\gamma B_0}{2\omega} (\sin \omega t) \right)$.

We can now ask: what is the minimum value of B_0 which will allow a probability of 1 for measuring S_x to be $-\hbar/2$? In other words, what is the minimum field that will guarantee a flip of the spin? We observe that this probability contains the term $\sin \omega t$ which oscillates between ± 1 . In order for the probability to reach 1, it must be possible for $\frac{\gamma B_0}{2\omega} (\sin \omega t)$ to reach $\pi/2$ at some time. Therefore we must have $\gamma B_0/2\omega = \pi/2$ as the lower limit, so the minimum magnetic field is

$$(15) \quad B_0(\min) = \frac{\pi\omega}{\gamma}$$