SPIN 1/2 PARTICLE IN TIME-VARYING MAGNETIC FIELD

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.33.

Here's an example of a time-dependent Hamiltonian in a spin 1/2 system. We saw in the last post that the energy of a particle with spin in a magnetic field is

$$H = -\gamma \mathbf{B}\cdot\mathbf{S} \tag{1}$$

Suppose now that the magnetic field is parallel to the z axis, but oscillates, so it is given by

$$\mathbf{B} = B_0 \cos(\omega t)\,\hat{\mathbf{k}} \tag{2}$$

The Hamiltonian for a spin 1/2 particle such as an electron is therefore $H = -\gamma B_0 \cos(\omega t)\, S_z$, or:

$$H = -\frac{\gamma B_0 \hbar}{2}\cos\omega t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3}$$

To find $\chi(t)$ we need to solve the time-dependent Schrödinger equation

$$i\hbar\frac{\partial \chi}{\partial t} = H\chi \tag{4}$$

If $\chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$, we get two differential equations

$$\begin{aligned}
\frac{\partial \chi_{1,2}}{\partial t} &= \pm i\frac{\gamma B_0}{2}(\cos\omega t)\chi_{1,2} &(5)\\
\frac{d\chi_{1,2}}{\chi_{1,2}} &= \pm i\frac{\gamma B_0}{2}(\cos\omega t)dt &(6)\\
\ln\chi_{1,2} &= \pm i\frac{\gamma B_0}{2\omega}(\sin\omega t)+\ln C_{1,2} &(7)\\
\chi_{1,2}(t) &= C_{1,2}e^{\pm i\gamma B_0(\sin\omega t)/2\omega} &(8)
\end{aligned}$$

where $C_{1,2}$ are constants that must be found from the initial condition. If we take the initial state of the electron to be spin up, then $\chi(0) = \chi_+^{(x)} =$

$\frac{1}{\sqrt{2}}\left(\begin{array}{c}1\\1\end{array}\right)$. From this condition we find $C_1 = C_2 = 1/\sqrt{2}$, so the final solution is:

$$\chi(t) = \frac{1}{\sqrt{2}}\left(\begin{array}{c} e^{i\gamma B_0(\sin\omega t)/2\omega} \\ e^{-i\gamma B_0(\sin\omega t)/2\omega}\end{array}\right) \tag{9}$$

To find the probability of getting $\hbar/2$ if we measure S_x, we can use the same method as in the last post, so we must express $\chi(t)$ in terms of the eigenspinors of S_x:

$$\chi(t) = \frac{1}{\sqrt{2}}\left(\begin{array}{c} e^{i\gamma B_0(\sin\omega t)/2\omega} \\ e^{-i\gamma B_0(\sin\omega t)/2\omega}\end{array}\right) = \frac{a}{\sqrt{2}}\left(\begin{array}{c}1\\1\end{array}\right) + \frac{b}{\sqrt{2}}\left(\begin{array}{c}1\\-1\end{array}\right) \tag{10}$$

from which we can solve for a and b:

$$a + b = e^{i\gamma B_0(\sin\omega t)/2\omega} \tag{11}$$
$$a - b = e^{-i\gamma B_0(\sin\omega t)/2\omega} \tag{12}$$
$$a = \cos\left(\frac{\gamma B_0}{2\omega}(\sin\omega t)\right) \tag{13}$$
$$b = i\sin\left(\frac{\gamma B_0}{2\omega}(\sin\omega t)\right) \tag{14}$$

The probability of a measurement of S_x giving $-\hbar/2$ is therefore $|b|^2 = \sin^2\left(\frac{\gamma B_0}{2\omega}(\sin\omega t)\right)$.

We can now ask: what is the minimum value of B_0 which will allow a probability of 1 for measuring S_x to be $-\hbar/2$? In other words, what is the minimum field that will guarantee a flip of the spin? We observe that this probability contains the term $\sin\omega t$ which oscillates between ± 1. In order for the probability to reach 1, it must be possible for $\frac{\gamma B_0}{2\omega}(\sin\omega t)$ to reach $\pi/2$ at some time. Therefore we must have $\gamma B_0/2\omega = \pi/2$ as the lower limit, so the minimum magnetic field is

$$B_0(min) = \frac{\pi\omega}{\gamma} \tag{15}$$