ANGULAR MOMENTUM: ADDING 2 SPINS

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Section 4.4.3 & Problem 4.34.

So far, every quantum system we've considered has consisted of a single particle in a potential (for spatial calculations) or a single particle with spin (as with the electron in the hydrogen atom). As a first look at dealing with more than one particle, we will examine the problem of the total spin of a hydrogen atom.

A hydrogen atom consists of a proton and an electron, both with spin 1/2. Classically, since angular momentum is a vector quantity, to obtain the total angular momentum of a system we just add the individual vectors. In quantum mechanics, we can start by following this path and see where it leads us.

The first point to make is that the proton and electron are independent particles; if we wished to describe their wave functions we would use separate sets of independent spatial variables. This is true also of their spins. The spin operator S_1 refers only to (say) the proton, while the operator S_2 refers to the electron. If we represent the spin state of the proton by χ_1 and that of the electron by χ_2, then each spin operator operates only on the spinor with the same suffix; it has no effect on the other spinor.

Since the two spins are independent, we require their operators to commute so that both spins are measurable at the same time.

We can write the combined state of the proton and electron as $\chi_1\chi_2$, where each of the χs can be written as a linear combination of the two eigenspinors of one of the components of the corresponding spin. For example, we might write χ_1 as a combination of the eigenspinors of S_{1z}:

$$\chi_1 = a_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{0.1}$$

It's important to understand that the eigenspinors serve as the basis for the spin space in much the same way as the three unit vectors serve as a basis for 3-d space. We can express the spatial coordinates of the proton and neutron in the same 3-d space, but the variables used to represent these two sets of coordinates are different and independent of each other. The same is true of the spin space.

The term $\chi_1\chi_2$ isn't really a product in the mathematical sense; rather it is a notation used to represent the compound state of the atom.

We can then define the total spin of the hydrogen atom as

$$\mathsf{S} = \mathsf{S}_1 + \mathsf{S}_2 \tag{0.2}$$

The z component of the total spin is just the sum of the z components of each spin separately, so we get

$$S_z\chi_1\chi_2 = (S_{1z} + S_{2z})\,\chi_1\chi_2 \tag{0.3}$$

$$= (S_{1z}\chi_1)\,\chi_2 + (S_{2z}\chi_2)\,\chi_1 \tag{0.4}$$

If each particle is in one of the eigenstates of S_z, then the last equation becomes

$$(S_{1z}\chi_1)\,\chi_2 + (S_{2z}\chi_2)\,\chi_1 = \hbar m_1\chi_1\chi_2 + \hbar m_2\chi_1\chi_2 \tag{0.5}$$

$$= (m_1 + m_2)\,\hbar\chi_1\chi_2 \tag{0.6}$$

Since the quantum number $m_{1,2} = \pm\frac{1}{2}$, the eigenvalues of the combined S_z are $-\hbar$, 0 and $\hbar$.

There is one little snag, however. If the combined spin state was really spin 1, we would expect these 3 eigenvalues for S_z, but we would expect only a single state for each eigenvalue. We have a single state for each of $\pm\hbar$, but we can get the 0 state either by having $m_1 = +\frac{1}{2}$, $m_2 = -\frac{1}{2}$ or $m_1 = -\frac{1}{2}$, $m_2 = +\frac{1}{2}$.

Are there really two states with $m = m_1 + m_2 = 0$? Let's start with the $m = -1$ state and apply the raising operator to it. The raising operators for the individual spins are

$$S_{1,2+} = S_{1.2x} + iS_{1,2y} \tag{0.7}$$

The compound raising operator is therefore

$$S_+ = S_{1x} + S_{2x} + \left(S_{1y} + S_{2y}\right) i = S_{1+} + S_{2+} \tag{0.8}$$

To make the notation a bit more graphic, we'll represent each state by a pair of arrows, with $\uparrow$ representing $m = +\frac{1}{2}$ (spin up) and $\downarrow$ $m = -\frac{1}{2}$ (spin down). We then have, for the single spin case

$$S_{1,2+}\downarrow = \hbar\uparrow \tag{0.9}$$

So we get

$$S_+(\downarrow\downarrow) = (S_{1+}+S_{2+})(\downarrow\downarrow) \tag{0.10}$$
$$= \hbar(\uparrow\downarrow+\downarrow\uparrow) \tag{0.11}$$

Thus the $m=0$ state we get here is one which is a combination of the two individual states. Thus the 3 states that correspond to a spin-1 state are

$$|1\,1\rangle = \uparrow\uparrow \tag{0.12}$$
$$|1\,0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow+\downarrow\uparrow) \tag{0.13}$$
$$|1\,-1\rangle = \downarrow\downarrow \tag{0.14}$$

The $1/\sqrt{2}$ in the second line is for normalization. These three states are referred to as the triplet state.

It's worth pointing out how you normalize compound states like this. Since the two spins are independent, the normalization works out as follows

(0.15)
$$(\uparrow\downarrow+\downarrow\uparrow)^T(\uparrow\downarrow+\downarrow\uparrow) = \left(\begin{bmatrix}1 & 0\end{bmatrix}_1\begin{bmatrix}0 & 1\end{bmatrix}_2+\begin{bmatrix}0 & 1\end{bmatrix}_1\begin{bmatrix}1 & 0\end{bmatrix}_2\right)\times$$
$$\left(\begin{bmatrix}1\\0\end{bmatrix}_1\begin{bmatrix}0\\1\end{bmatrix}_2+\begin{bmatrix}0\\1\end{bmatrix}_1\begin{bmatrix}1\\0\end{bmatrix}_2\right) \tag{0.16}$$

The matrix products occur only between matrices with the same subscript, so, for example

(0.17)
$$\begin{bmatrix}1 & 0\end{bmatrix}_1\begin{bmatrix}0 & 1\end{bmatrix}_2\begin{bmatrix}1\\0\end{bmatrix}_1\begin{bmatrix}0\\1\end{bmatrix}_2 = \left(\begin{bmatrix}1 & 0\end{bmatrix}_1\begin{bmatrix}1\\0\end{bmatrix}_1\right)\left(\begin{bmatrix}0 & 1\end{bmatrix}_2\begin{bmatrix}0\\1\end{bmatrix}_2\right) = 1\times 1 = 1$$

Using similar calculations for the other products yields

$$(\uparrow\downarrow+\downarrow\uparrow)^T(\uparrow\downarrow+\downarrow\uparrow) = 2 \tag{0.18}$$

hence the normalization factor of $1/\sqrt{2}$. A similar calculation shows the three states above are mutually orthogonal. That is, for example $\langle 1\,1|\,1\,0\rangle = 0$, etc.

To confirm that these states really do belong to a spin-1 system, we need to show that they are eigenstates of the total spin operator S^2 with eigenvalue $\hbar^2 s(s+1) = 2\hbar^2$. In order to do this, we need to work out this operator. We have

$$S^2 = S_1^2 + S_2^2 + 2\mathsf{S}_1 \cdot \mathsf{S}_2 \tag{0.19}$$

The first two terms are easy enough, and the last term is

$$2\mathsf{S}_1 \cdot \mathsf{S}_2 = 2\left(S_{1x}S_{2x} + S_{1y}S_{2y} + S_{1z}S_{2z}\right) \tag{0.20}$$

From our analysis of spin 1/2 earlier, we have

$$S_x(\uparrow) = \frac{\hbar}{2}\downarrow \tag{0.21}$$

$$S_x(\downarrow) = \frac{\hbar}{2}\uparrow \tag{0.22}$$

$$S_y(\uparrow) = i\frac{\hbar}{2}\downarrow \tag{0.23}$$

$$S_y(\downarrow) = -i\frac{\hbar}{2}\uparrow \tag{0.24}$$

$$S_z(\uparrow) = \frac{\hbar}{2}\uparrow \tag{0.25}$$

$$S_z(\downarrow) = -\frac{\hbar}{2}\downarrow \tag{0.26}$$

Applying this to the states $\uparrow\uparrow$ and $\downarrow\downarrow$, we get

$$S^2|1\,1\rangle = ((S_1)^2 + (S_2)^2 + 2\mathsf{S}_1 \cdot \mathsf{S}_2)(\uparrow\uparrow) \tag{0.27}$$

$$= \left(\frac{3\hbar^2}{4} + \frac{3\hbar^2}{4}\right)(\uparrow\uparrow) + \left(\frac{2\hbar^2}{4} - \frac{2\hbar^2}{4}\right)(\downarrow\downarrow) + \frac{2\hbar^2}{4}(\uparrow\uparrow) \tag{0.28}$$

$$= 2\hbar^2|1\,1\rangle \tag{0.29}$$

$$S^2|1\,-1\rangle = ((S_1)^2 + (S_2)^2 + 2\mathsf{S}_1 \cdot \mathsf{S}_2)(\downarrow\downarrow) \tag{0.30}$$

$$= \left(\frac{3\hbar^2}{4} + \frac{3\hbar^2}{4}\right)(\downarrow\downarrow) + \left(\frac{2\hbar^2}{4} - \frac{2\hbar^2}{4}\right)(\uparrow\uparrow) + \frac{2\hbar^2}{4}(\downarrow\downarrow) \tag{0.31}$$

$$= 2\hbar^2|1\,-1\rangle \tag{0.32}$$

A similar calculation gives

$$S^2|1\,0\rangle = 2\hbar^2|1\,0\rangle \tag{0.33}$$

What of the other $m = 0$ state? We can construct a state that is orthogonal to the three states above, and that has $m = 0$, as follows.

$$|0\,0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow) \tag{0.34}$$

We've written this as a spin 0 state, and doing a similar calculation to that above does indeed yield $S^2\,|0\,0\rangle = 0$.

We can check the actions of the raising and lowering operators on various states.

$$\begin{aligned} S_-|1\,0\rangle &= \frac{1}{\sqrt{2}}S_-(\uparrow\downarrow + \downarrow\uparrow) && (0.35)\\ &= \frac{1}{\sqrt{2}}(S_{1-} + S_{2-})(\uparrow\downarrow + \downarrow\uparrow) && (0.36)\\ &= \frac{\hbar}{\sqrt{2}}(\downarrow\downarrow + \downarrow\downarrow) && (0.37)\\ &= \sqrt{2}\hbar|1\,-1\rangle && (0.38) \end{aligned}$$

using $S_{j-}\downarrow = 0$ for $j = 1,2$ (that is, applying a lowering operator to a bottom state gives zero).

$$\begin{aligned} S_+|0\,0\rangle &= \frac{1}{\sqrt{2}}S_+(\uparrow\downarrow - \downarrow\uparrow) && (0.39)\\ &= \frac{1}{\sqrt{2}}(S_{1+} + S_{2+})(\uparrow\downarrow - \downarrow\uparrow) && (0.40)\\ &= \frac{\hbar}{\sqrt{2}}(\uparrow\uparrow - \uparrow\uparrow) && (0.41)\\ &= 0 && (0.42)\\ S_-|0\,0\rangle &= \frac{1}{\sqrt{2}}S_-(\uparrow\downarrow - \downarrow\uparrow) && (0.43)\\ &= \frac{1}{\sqrt{2}}(S_{1-} + S_{2-})(\uparrow\downarrow - \downarrow\uparrow) && (0.44)\\ &= \frac{\hbar}{\sqrt{2}}(\downarrow\downarrow - \downarrow\downarrow) && (0.45)\\ &= 0 && (0.46) \end{aligned}$$

Thus both the raising and lowering operators acting on the singlet state give zero, as we would expect for a spin-0 state.

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