

## ANGULAR MOMENTUM: ADDITION AND CLEBSCH-GORDAN COEFFICIENTS

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.36.

When we analyzed the addition of two spin 1/2 angular momenta, we found that there were two states with  $m = 0$ , and that we could form two orthogonal combinations of states to get states that were eigenvectors of  $S^2$ . These states were

$$(0.1) \quad |10\rangle = \frac{1}{\sqrt{2}} \left( \left| \frac{1}{2} \frac{1}{2} \right\rangle \left| \frac{1}{2} - \frac{1}{2} \right\rangle + \left| \frac{1}{2} - \frac{1}{2} \right\rangle \left| \frac{1}{2} \frac{1}{2} \right\rangle \right)$$

$$(0.2) \quad |00\rangle = \frac{1}{\sqrt{2}} \left( \left| \frac{1}{2} \frac{1}{2} \right\rangle \left| \frac{1}{2} - \frac{1}{2} \right\rangle - \left| \frac{1}{2} - \frac{1}{2} \right\rangle \left| \frac{1}{2} \frac{1}{2} \right\rangle \right)$$

The first state has  $s = 1$  and the second,  $s = 0$ . In each case, the  $1/\sqrt{2}$  is for normalization.

We can generalize the solution by finding which linear combination of states of the constituent spins produce each of the possible total spin states. In general (although a proof would take us too far afield), if we combine two states with spins (or, indeed, any angular momenta, spin or orbital)  $s_1$  and  $s_2$ , we can produce states with any value of total spin from  $|s_1 - s_2|$  up to  $s_1 + s_2$ , in steps of 1. For example, combining spin states  $s_1 = 3/2$  with  $s_2 = 1$  produces possible states from  $s = s_1 - s_2 = 1/2$  and then  $s = 3/2$ , finishing with  $s = s_1 + s_2 = 5/2$ .

If we specify the values of  $s$  and  $m$  for the compound state, then we can form this compound state out of a linear combination of the constituent states just as we did above for adding two spin 1/2 states. The challenge is in finding the precise linear combinations that are also eigenstates of  $S^2$ .

For example, with the spin 1/2 system, a *possible* combined state is

$$(0.3) \quad \frac{1}{\sqrt{3}} \left| \frac{1}{2} \frac{1}{2} \right\rangle \left| \frac{1}{2} - \frac{1}{2} \right\rangle + \sqrt{\frac{2}{3}} \left| \frac{1}{2} - \frac{1}{2} \right\rangle \left| \frac{1}{2} \frac{1}{2} \right\rangle$$

Although this state is normalized, it is *not* an eigenstate of  $S^2$ , as can be verified by direct calculation. The total spin squared operator is  $S_1^2 + S_2^2 +$

$2S_1 \cdot S_2$  and applying it to this state gives (using the results from the last post):

(0.4)

$$(S_1^2 + S_2^2 + 2S_1 \cdot S_2) \left| \frac{1}{2} \frac{1}{2} \right\rangle \left| \frac{1}{2} - \frac{1}{2} \right\rangle = \left( \frac{3\hbar^2}{4} + \frac{3\hbar^2}{4} \right) \left| \frac{1}{2} \frac{1}{2} \right\rangle \left| \frac{1}{2} - \frac{1}{2} \right\rangle +$$

$$(0.5) \quad \hbar^2 \left| \frac{1}{2} - \frac{1}{2} \right\rangle \left| \frac{1}{2} \frac{1}{2} \right\rangle - \frac{2\hbar^2}{4} \left| \frac{1}{2} \frac{1}{2} \right\rangle \left| \frac{1}{2} - \frac{1}{2} \right\rangle$$

$$(0.6) \quad = \hbar^2 \left( \left| \frac{1}{2} \frac{1}{2} \right\rangle \left| \frac{1}{2} - \frac{1}{2} \right\rangle + \left| \frac{1}{2} - \frac{1}{2} \right\rangle \left| \frac{1}{2} \frac{1}{2} \right\rangle \right)$$

(0.7)

$$(S_1^2 + S_2^2 + 2S_1 \cdot S_2) \left| \frac{1}{2} - \frac{1}{2} \right\rangle \left| \frac{1}{2} \frac{1}{2} \right\rangle = \left( \frac{3\hbar^2}{4} + \frac{3\hbar^2}{4} \right) \left| \frac{1}{2} - \frac{1}{2} \right\rangle \left| \frac{1}{2} \frac{1}{2} \right\rangle +$$

$$(0.8) \quad \hbar^2 \left| \frac{1}{2} \frac{1}{2} \right\rangle \left| \frac{1}{2} - \frac{1}{2} \right\rangle - \frac{2\hbar^2}{4} \left| \frac{1}{2} - \frac{1}{2} \right\rangle \left| \frac{1}{2} \frac{1}{2} \right\rangle$$

$$(0.9) \quad = \hbar^2 \left( \left| \frac{1}{2} \frac{1}{2} \right\rangle \left| \frac{1}{2} - \frac{1}{2} \right\rangle + \left| \frac{1}{2} - \frac{1}{2} \right\rangle \left| \frac{1}{2} \frac{1}{2} \right\rangle \right)$$

Substituting back into 0.3, we get

$$(0.10) \quad S^2 \left( \frac{1}{\sqrt{3}} \left| \frac{1}{2} \frac{1}{2} \right\rangle \left| \frac{1}{2} - \frac{1}{2} \right\rangle + \sqrt{\frac{2}{3}} \left| \frac{1}{2} - \frac{1}{2} \right\rangle \left| \frac{1}{2} \frac{1}{2} \right\rangle \right) =$$

$$(0.11) \quad \frac{2(1+\sqrt{2})}{\sqrt{3}} \hbar^2 \left( \left| \frac{1}{2} \frac{1}{2} \right\rangle \left| \frac{1}{2} - \frac{1}{2} \right\rangle + \left| \frac{1}{2} - \frac{1}{2} \right\rangle \left| \frac{1}{2} \frac{1}{2} \right\rangle \right)$$

The action of applying  $S^2$  to this state results in a state where both terms have the same coefficient, so it's not just a multiple of the original state, where the coefficients are different.

In general, then, the problem is to find the coefficients of the individual states that contribute to a compound state. That is, we're looking for the coefficients in the sum

$$(0.12) \quad |s m\rangle = \sum_{m_1+m_2=m} C_{m_1 m_2 m}^{s_1 s_2 s} |s_1 m_1\rangle |s_2 m_2\rangle$$

where we would specify the values of  $s_1, s_2, s$  and  $m$ . The coefficients are known as Clebsch-Gordan coefficients, and their calculation is a bit tricky in the general case. For a general formula (be warned: it's not pretty) and tables of the coefficients, I refer the reader to this Wikipedia article.

To use the tables, refer to the case of adding two spin 1/2s above (the tables use the symbol  $j$  instead of  $s$ ). Look at the tables for  $j_1 = 1/2$ ,  $j_2 = 1/2$ . The first table gives the coefficient for forming the final state  $|11\rangle$ . You can see that this state can be formed using only the  $m_1 = 1/2$ ,  $m_2 = 1/2$  state, and that the coefficient is 1.

In the second table, we see how to form the states with  $m = 0$ . There are, as we've seen, two possibilities:  $j = 1$  or  $j = 0$ . In the former case, we use the first column of coefficients, and we get the  $|10\rangle$  state above. For  $j = 0$ , we use the second column and we get the  $|00\rangle$  state above. The probability of finding the system in one of the states in the sum is, as usual, the square of the coefficient of that state.

To get the coefficients for negative  $m$ , just look at the corresponding table for positive  $m$  and flip the signs of  $m_1$  and  $m_2$ . The coefficients remain the same.

As another example, suppose a spin-1 particle and a spin-2 particle are at rest (so there is no orbital angular momentum), and are in a compound state with spin 3 and  $z$  component  $\hbar$ . From the Clebsch-Gordan table, look at the entry under  $j_1 = 2$ ,  $j_2 = 1$  for  $j = 3$ ,  $m = 1$ . We see that the possible values of  $m_1$  (the  $z$ -component of the spin 2 particle) are +2 (prob  $(1/\sqrt{15})^2 = 1/15$ ), +1 (prob  $8/15$ ) and 0 (prob  $2/5$ ).

As an example of combining spin and orbital angular momentum, consider an electron in the state  $\psi_{510}$  of the hydrogen atom. If the electron has spin down, what are the possible values of total angular momentum of the electron alone?

In state  $\psi_{510}$  the orbital angular momentum is  $l = 1$ ,  $m_1 = 0$ . An electron with spin down has spin  $s = 1/2$ ,  $m_2 = -1/2$ . From the Clebsch-Gordan table, we look under  $j_1 = 1$ ,  $j_2 = 1/2$  for  $m_1 = 0$  and  $m_2 = -1/2$ . This time, we know the  $m$  values for each component of the angular momentum separately and need to know what possible total values there are. To get this, we first determine the total  $z$  component, which is  $m = m_1 + m_2 = -\frac{1}{2}$ . Since this is negative, we look at the table for  $m = +\frac{1}{2}$  and flip the signs of  $m_1$  and  $m_2$ . Thus the row we want is the 0, 1/2 row. Reading across the row, we get the possible values for the  $j$  as  $3/2$  and  $1/2$  so the possible values for total angular momentum squared are  $j(j+1)\hbar^2$ , which are  $15\hbar^2/4$  (prob  $2/3$ ) and  $3\hbar^2/4$  (prob  $1/3$ ).

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