

ANGULAR MOMENTUM: COMMUTATORS OF ADDED SPINS

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.37.

When discussing the general addition of two spins, we saw that only certain linear combinations of the constituent states are eigenstates of S^2 . Here, we'll have a look at some commutator relations that are relevant to this. Let's examine the commutator of the total spin squared S^2 with the z component of one of the individual spins S_{1z} .

The total spin is $S = \mathbf{S}_1 + \mathbf{S}_2$. Since the spin operators \mathbf{S}_1 and \mathbf{S}_2 operate on different spins, any component of one commutes with any component of the other. Also, since S_{ij}^2 (for $i = 1, 2$ and $j = x, y, z$) is proportional to the identity matrix, it commutes with everything. Using these facts to simplify the expression, we get:

$$\begin{aligned}(0.1) \quad [S^2, S_{1z}] &= (\mathbf{S}_1 + \mathbf{S}_2)^2 S_{1z} - S_{1z} (\mathbf{S}_1 + \mathbf{S}_2)^2 \\(0.2) &= 2(S_{1x}S_{1z}S_{2x} + S_{1y}S_{1z}S_{2y} - S_{1z}S_{1x}S_{2x} - S_{1z}S_{1y}S_{2y}) \\(0.3) &= 2[S_{1x}, S_{1z}]S_{2x} + 2[S_{1y}, S_{1z}]S_{2y} \\(0.4) &= -2i\hbar S_{1y}S_{2x} + 2i\hbar S_{1x}S_{2y} \\(0.5) &= 2i\hbar [\mathbf{S}_1 \times \mathbf{S}_2]_z\end{aligned}$$

where $[\mathbf{S}_1 \times \mathbf{S}_2]_z$ indicates the third or z component of $\mathbf{S}_1 \times \mathbf{S}_2$.

A pair of similar calculations gives the other two components to give the general relation:

$$(0.6) \quad [S^2, \mathbf{S}_1] = 2i\hbar \mathbf{S}_1 \times \mathbf{S}_2$$

The fact that the commutator of S^2 with each of the components of the two spins making up the total spin is non-zero means that we cannot measure the total spin and the component of any of the individual spins at the same time. We therefore need a linear combination of the individual spin states to give an observable compatible with S^2 . As we saw in the last post, however, we can't take just any linear combination, since only certain ones are eigenstates of S^2 .

However, in the special case where \mathbf{S}_1 is equal to \mathbf{S}_2 , the cross product is zero and the commutator is then zero. This happens when, for example in the spin $1/2$ system, both spins are spin up or spin down, and in that case, as we saw earlier, the compound spin state is just a single term.