ANGULAR MOMENTUM: COMMUTATORS OF ADDED SPINS

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.37.

When discussing the general addition of two spins, we saw that only certain linear combinations of the constituent states are eigenstates of S^2. Here, we'll have a look at some commutator relations that are relevant to this. Let's examine the commutator of the total spin squared S^2 with the z component of one of the individual spins S_{1z}.

The total spin is $S = \mathbf{S}_1 + \mathbf{S}_2$. Since the spin operators $\mathbf{S}_1$ and $\mathbf{S}_2$ operate on different spins, any component of one commutes with any component of the other. Also, since S_{ij}^2 (for $i = 1,2$ and $j = x,y,z$) is proportional to the identity matrix, it commutes with everything. Using these facts to simplify the expression, we get:

$$\begin{aligned}
(1)\qquad [S^2, S_{1z}] &= (\mathbf{S}_1 + \mathbf{S}_2)^2 S_{1z} - S_{1z}(\mathbf{S}_1 + \mathbf{S}_2)^2 \\
(2)\qquad &= 2(S_{1x}S_{1z}S_{2x} + S_{1y}S_{1z}S_{2y} - S_{1z}S_{1x}S_{2x} - S_{1z}S_{1y}S_{2y} \\
(3)\qquad &= 2[S_{1x}, S_{1z}]S_{2x} + 2[S_{1y}, S_{1z}]S_{2y} \\
(4)\qquad &= -2i\hbar S_{1y}S_{2x} + 2i\hbar S_{1x}S_{2y} \\
(5)\qquad &= 2i\hbar [\mathbf{S}_1 \times \mathbf{S}_2]_z
\end{aligned}$$

where $[\mathbf{S}_1 \times \mathbf{S}_2]_z$ indicates the third or z component of $\mathbf{S}_1 \times \mathbf{S}_2$.

A pair of similar calculations gives the other two components to give the general relation:

$$[S^2, \mathbf{S}_1] = 2i\hbar \mathbf{S}_1 \times \mathbf{S}_2 \qquad (6)$$

The fact that the commutator of S^2 with each of the components of the two spins making up the total spin is non-zero means that we cannot measure the total spin and the component of any of the individual spins at the same time. We therefore need a linear combination of the individual spin states to give an observable compatible with S^2. As we saw in the last post, however, we can't take just any linear combination, since only certain ones are eigenstates of S^2.

However, in the special case where $\mathbf{S}_1$ is equal to $\mathbf{S}_2$, the cross product is zero and the commutator is then zero. This happens when, for example in the spin 1/2 system, both spins are spin up or spin down, and in that case, as we saw earlier, the compound spin state is just a single term.