HARMONIC OSCILLATOR IN 3-D - RECTANGULAR COORDINATES

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.38.

The 3-d harmonic oscillator can be solved in rectangular coordinates by separation of variables. The Schrödinger equation to be solved for the 3-d harmonic oscillator is

$$-\frac{\hbar}{2m}\nabla^2\psi+\frac{1}{2}m\omega^2(x^2+y^2+z^2)\psi=E\psi \tag{1}$$

To use separation of variables we define

$$\psi(x,y,z)=\xi(x)\eta(y)\zeta(z) \tag{2}$$

Dividing 1 through by this product we get

$$-\frac{\hbar^2}{2m}\frac{\xi''}{\xi}+\frac{1}{2}m\omega^2x^2-\frac{\hbar^2}{2m}\frac{\eta''}{\eta}+\frac{1}{2}m\omega^2y^2-\frac{\hbar^2}{2m}\frac{\zeta''}{\zeta}+\frac{1}{2}m\omega^2z^2=E \tag{3}$$

where the double prime notation indicates the second derivative of a function with respect to its independent variable, so $\xi''=d^2\xi/dx^2$, etc.

We now have three groups of two terms each of which depends on only one of the variables x, y and z, and the sum of all these terms is the constant E. We can therefore use the usual argument that each group of two terms must be a constant on its own, so the 3-d equation reduces to the sum of three 1-d harmonic oscillators. From the analysis of the 1-d harmonic oscillator, we know that each of these will contribute $(n+1/2)\hbar\omega$ to the total energy, with the ground state at $n=0$. Thus the ground state for the 3-d oscillator will have energy $3\hbar\omega/2$, and the general energy level will increase in steps of $\hbar\omega$ so the energy levels are given by

$$E_n=\left(n+\frac{3}{2}\right)\hbar\omega \tag{4}$$

Unlike the 1-d case, the energies of the 3-d oscillator are degenerate. A given value of n is composed of the sum of 3 quantum numbers: $n=n_x+$

$n_y + n_z$ where all numbers are non-negative integers. Suppose we choose a value for n_x so that $n_y + n_z = n - n_x$. The number of pairs of integers that can be used for $n_y + n_z$ is $n - n_x + 1$ (since n_y can be anything between 0 and $n - n_x$). Since n_x itself can range between 0 and n, the total number of combinations of quantum states that can make up state n is

$$d(n) = \sum_{n_x=0}^{n} (n - n_x + 1) \tag{5}$$

$$= (n+1)\sum_{n_x=0}^{n} 1 - \sum_{n_x=0}^{n} n_x \tag{6}$$

$$= (n+1)^2 - \frac{1}{2}n(n+1) \tag{7}$$

$$= \frac{1}{2}(n+1)(n+2) \tag{8}$$

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