

## HYDROGEN ATOM - COMPLETE WAVE FUNCTION

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.43.

If we combine all the results from the solution of the angular and radial equations for the hydrogen atom, we get a formula for the spatial wave function, which is given in Griffiths's book as eqn 4.89:

$$\psi_{nlm} = \sqrt{\left(\frac{2}{na}\right)^3 \frac{(n-l-1)!}{2n[(n+l)!]^3}} e^{-r/na} \left(\frac{2r}{na}\right)^l L_{n-l-1}^{2l+1}\left(\frac{2r}{na}\right) Y_l^m(\theta, \phi) \quad (1)$$

where  $L$  is an associated Laguerre polynomial and  $Y$  is a spherical harmonic.

The spherical harmonics can be calculated from a standard formula, as can the associated Laguerre polynomials. The forms of these functions vary according to the normalization. My version is

$$L_p^q(x) = c_0 \sum_{j=0}^p \frac{(-1)^j (p+q)!}{(p-j)!(q+j)!j!} x^j \quad (2)$$

The version in Griffiths sets  $c_0 = (p+q)!$  (so it is that version that is used in the above formula for  $\psi_{nlm}$ ) while some other sources use  $c_0 = 1$ .

For the spherical harmonics, the formulas are

$$Y_l^m(\theta, \phi) = \left[ \frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!} \right]^{1/2} e^{im\phi} P_l^m(\cos\theta) \quad (3)$$

with the  $P_l^m$  being the associated Legendre function:

$$P_l^m(x) = (1-x^2)^{m/2} \sum_{k=0}^{[(l-m)/2]} \frac{(2l-2k)!}{2^l(l-k)!k!(l-2k-m)!} (-1)^k x^{l-m-2k} \quad (4)$$

As an example of using this formula, we'll construct  $\psi_{321}$ . We get  $L_0^5(x) = 120$  and we worked out  $Y_2^1$  in the previous post as

$$Y_2^1(\theta, \phi) = -\sqrt{\frac{15}{8\pi}} e^{i\phi} \sin\theta \cos\theta \quad (5)$$

Plugging these into the overall formula gives

$$\psi_{321} = -\sqrt{\left(\frac{2}{3a}\right)^3 \frac{1}{6 \times 120^3}} e^{-r/3a} \left(\frac{2r}{3a}\right)^2 (120) \left(\frac{15}{8\pi}\right)^{\frac{1}{2}} \sin\theta \cos\theta e^{i\phi} \quad (6)$$

$$= -\frac{1}{81\sqrt{\pi a^7}} r^2 e^{-r/3a} \sin\theta \cos\theta e^{i\phi} \quad (7)$$

To check the normalization, we do the integral:

$$\int_0^{2\pi} \int_0^\pi \int_0^\infty |\psi_{321}|^2 r^2 \sin\theta dr d\theta d\phi = \frac{1}{81^2 \pi a^7} \int_0^{2\pi} \int_0^\pi \int_0^\infty r^6 e^{-2r/3a} \sin^3\theta \cos^2\theta dr d\theta d\phi \quad (8)$$

$$= 1 \quad (9)$$

using Maple for the integral.

The expectation value of  $r^s$  is (using Maple)

$$\langle r^s \rangle = \int_0^{2\pi} \int_0^\pi \int_0^\infty |\psi_{321}|^2 r^{2+s} \sin\theta dr d\theta d\phi \quad (10)$$

$$= \frac{1}{81^2 \pi a^7} \int_0^{2\pi} \int_0^\pi \int_0^\infty r^{6+s} e^{-2r/3a} \sin^3\theta \cos^2\theta dr d\theta d\phi \quad (11)$$

$$= \frac{1}{720} \left(\frac{3a}{2}\right)^s \Gamma(7+s) \quad (12)$$

The gamma function is infinite at all non-positive integral arguments, so this value is finite for  $s > -7$  and all non-integer values less than  $-7$ . The smallest integer value for which this is finite is  $s = -6$ . Note in particular that if  $s = 0$ , the formula reduces to  $\langle r^0 \rangle = 1$  as it should.

The expectation value of  $r$  itself is

$$\langle r \rangle = \frac{21a}{2} \quad (13)$$

This is quite a bit bigger than the value for the ground state, which is  $\langle r \rangle = 3a/2$ .

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