

### HYDROGEN ATOM: WAVE FUNCTION EXAMPLE 3

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.44.

Another example of the hydrogen atom wave function. This time, we'll look at  $\psi_{433}$ . Using the formulas in the last post we get  $L_0^7(x) = 7! = 5040$ , and

$$(1) \quad Y_3^3 = -\left(\frac{35}{64\pi}\right)^{\frac{1}{2}} \sin^3 \theta e^{3i\phi}$$

The wave function is then:

$$(2) \quad \psi_{433} = \sqrt{\left(\frac{1}{2a}\right)^3 \frac{1}{6 \times 5040^3}} e^{-r/4a} \left(\frac{r}{2a}\right)^3 (5040) \left(-\left(\frac{35}{64\pi}\right)^{\frac{1}{2}} \sin^3 \theta e^{3i\phi}\right)$$

$$(3) \quad = -\frac{1}{6144\sqrt{\pi a^9}} r^3 e^{-r/4a} \sin^3 \theta e^{3i\phi}$$

As a check, we verify that this function is normalized:

$$(4) \quad \int_0^{2\pi} \int_0^\pi \int_0^\infty |\psi_{433}|^2 r^2 \sin \theta dr d\theta d\phi = \frac{1}{6144^2 \pi a^9} \int_0^{2\pi} \int_0^\pi \int_0^\infty r^8 e^{-r/2a} \sin^7 \theta dr d\theta d\phi$$
$$(5) \quad = 1$$

using Maple for the integral.

The expectation value of  $r$  is

$$(6) \quad \langle r \rangle = \int_0^{2\pi} \int_0^\pi \int_0^\infty |\psi_{433}|^2 r^3 \sin \theta dr d\theta d\phi$$

$$(7) \quad = \frac{1}{6144^2 \pi a^9} \int_0^{2\pi} \int_0^\pi \int_0^\infty r^9 e^{-r/2a} \sin^7 \theta dr d\theta d\phi$$

$$(8) \quad = 18a$$

As in the previous post, we can generalize this to find the expectation value of  $r^s$ :

$$(9) \quad \langle r^s \rangle = \frac{1}{40320} (2a)^s \Gamma(9+s)$$

The gamma function is infinite at all non-positive integral arguments, so this value is finite for  $s > -9$  and all non-integer values less than  $-9$ . The smallest integer value for which this is finite is  $s = -8$ . Note in particular that if  $s = 0$ , the formula reduces to  $\langle r^0 \rangle = \Gamma(9)/40320 = 8!/40320 = 1$  as it should.

Since  $\psi_{433}$  is an eigenfunction of  $L^2$  with eigenvalue  $l(l+1)\hbar^2 = 12\hbar^2$  and is also an eigenfunction of  $L_z$  with eigenvalue  $m\hbar = 3\hbar$  the only possible value of  $L_x^2 + L_y^2 = L^2 - L_z^2$  is  $(12 - 9)\hbar^2 = 3\hbar^2$ .