HYDROGEN ATOM: WAVE FUNCTION EXAMPLE 3

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 4.44.

Another example of the hydrogen atom wave function. This time, we'll look at ψ_{433}. Using the formulas in the last post we get $L_0^7(x) = 7! = 5040$, and

$$Y_3^3 = -\left(\frac{35}{64\pi}\right)^{\frac{1}{2}} \sin^3\theta e^{3i\phi} \tag{1}$$

The wave function is then:

$$\psi_{433} = \sqrt{\left(\frac{1}{2a}\right)^3 \frac{1}{6\times 5040^3}} e^{-r/4a} \left(\frac{r}{2a}\right)^3 (5040)\left(-\left(\frac{35}{64\pi}\right)^{\frac{1}{2}} \sin^3\theta e^{3i\phi}\right) \tag{2}$$

$$= -\frac{1}{6144\sqrt{\pi a^9}} r^3 e^{-r/4a} \sin^3\theta e^{3i\phi} \tag{3}$$

As a check, we verify that this function is normalized:

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} |\psi_{433}|^2 r^2 \sin\theta dr d\theta d\phi = \frac{1}{6144^2\pi a^9}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} r^8 e^{-r/2a}\sin^7\theta dr d\theta d\phi \tag{4}$$

$$= 1 \tag{5}$$

using Maple for the integral.

The expectation value of r is

$$\langle r\rangle = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} |\psi_{433}|^2 r^3 \sin\theta dr d\theta d\phi \tag{6}$$

$$= \frac{1}{6144^2\pi a^9}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} r^9 e^{-r/2a}\sin^7\theta dr d\theta d\phi \tag{7}$$

$$= 18a \tag{8}$$

As in the previous post, we can generalize this to find the expectation value of r^s:

$$\left\langle r^s\right\rangle = \frac{1}{40320}(2a)^s\Gamma(9+s) \tag{9}$$

The gamma function is infinite at all non-positive integral arguments, so this value is finite for $s>-9$ and all non-integer values less than -9. The smallest integer value for which this is finite is $s=-8$. Note in particular that if $s=0$, the formula reduces to $\left\langle r^0\right\rangle = \Gamma(9)/40320 = 8!/40320 = 1$ as it should.

Since ψ_{433} is an eigenfunction of L^2 with eigenvalue $l(l+1)\hbar^2 = 12\hbar^2$ and is also an eigenfunction of L_z with eigenvalue $m\hbar = 3\hbar$ the only possible value of $L_x^2+L_y^2 = L^2-L_z^2$ is $(12-9)\hbar^2 = 3\hbar^2$.